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Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work

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Abstract
The last decade has seen the development in France of a significant body of research into the teaching and learning of mathematics in CAS environments. As part of this, French researchers have reflected on issues of 'instrumentation', and the dialectics between conceptual and technical work in mathematics. The reflection presented here is more than a personal one — it is based on the collaboration and dialogues that I have been involved in during the Nineties. After a short introduction, I briefly present the main theoretical frameworks which we have used and developed in the French research: the anthropological approach in didactics initiated by Chevallard, and the theory of instrumentation developed in cognitive ergonomics. Turning to the CAS research, I show how these frameworks have allowed us to approach important issues as regards the educational use of CAS technology, focusing on the following points: the unexpected complexity of instrumental genesis, the mathematical needs of instrumentation, the status of instrumented techniques, the problems arising from their connection with paper/pencil techniques, and their institutional management.

Keywords: *Learning Mathematics, Genesis of a Reflection, Instrumentation, Technical and Conceptual Work*

1.0 INTRODUCTION

The development of mathematics has always been dependent upon the material and symbolic tools available for mathematical computations. Nobody would deny the role played by the introduction of the decimal system, the construction of logarithmic tables, the tabulation of elementary functions, or the development of mechanical and graphical computational tools. Today, advances in computations are linked to the development of numerical and symbolic mathematical software, among which Computer Algebra Systems (CAS in the following) play an increasing role. Professional mathematicians and engineers know that these sophisticated new tools don't become immediately efficient mathematical instruments for the user: their complexity does not make it easy to master, and fully benefit from, their potential. Professionals accept that there is a cost to learning how to effectively use such software. They also know that these tools have progressively changed their mathematical practices and, for some of them, even the "problématique" of their mathematical work. The necessity of research linked to the development of software packages, is today completely recognised as a specific area of mathematical research.

In the educational world, except for advanced university courses and professional training, the dominant vision contrasts with that of the professionals. What is aimed at by mathematics education, and especially by general mathematics education in school and university, is not an efficient mathematical practice, assisted by the currently available computational tools; rather, it is concerned with the transmission of the bases of 'mathematical culture'. The values of such a culture are social values and, like any social values, they have a stable core which tends to shape our relationships with and interpretation of the surrounding world (Abric, 1987). These values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practices linked to technological evolution. What is firstly asked of software and computational tools is to be pedagogical instruments for the learning of mathematical knowledge and values which were defined in the past, mostly before these tools existed. The tools are also put forward to help in the fight against "inadequate" teaching practices: practices too much orientated towards pure lecturing or the procedural learning of mathematical skills (if not to tackle the difficulties in schools induced by more general social problems). Under these conditions, it is especially difficult for mathematics educators to avoid ideological traps, and to deal with the issues of computational instrumentation, of relationships between technical and conceptual learning, and between paper/pencil and "instrumented"1 techniques, in a sensible way.

In this paper, I want to contribute to the mathematics education community's reflection on these issues. I will do so by relying on the results of different research projects which, in my opinion, constitute a coherent set focusing on these issues. I will firstly present the theoretical frameworks they have used and contributed to. I will then try to

[1] Of course, any mathematical technique is in some way an instrumented technique but here I reserve this term to techniques instrumented by computer tools, especially CAS.

present what I see as the major contributions of these research projects. Of course, this research work doesn't pretend to cover all the issues linked to the educational use of CAS but, in my opinion, it has the merit of attracting our attention to important issues the educational literature on CAS has not been very sensitive to, up to now, and to provide some conceptual tools in order to tackle these.

The research work which I will refer to here is mainly French research carried out by different teams in Paris, Rennes and Montpellier, since 1993 (Artigue, 1997; Artigue & al., 1997; Guin & Delgoulet, 1997; Guin & Trouche, 1999, 2002; Lagrange, 1999, 2001; Trouche, 1997, 2000; Defouad, 2000). These projects cannot be considered as independent, and I have been personally involved in some of them. It has been through such mutual interaction, more and more active as time has passed, that our reflection on instrumental issues has developed and matured. For that reason, I will often speak of "we" in the following. But I would like to emphasise that the vision I have of this genesis is certainly a personal one, and that the interpretations presented here are my sole responsibility.

2.0 A THEORETICAL FRAMEWORK FOR THINKING ABOUT LEARNING ISSUES IN CAS ENVIRONMENTS

Our conviction that the theoretical frameworks structuring research on CAS technology (see for instance (Kutzler, 1994)) were not necessarily the most adequate, emerged from a first research work carried out as part of a national project on CAS integration at secondary level and in CPGE classes[2], which began in the early Nineties (Hirlimann & al., 1996), (Artigue, 1997). In the mid-nineties, we thus became increasingly aware of the fact that we needed other frameworks in order to overcome some research traps that we were more and more sensitive to, the first one being what we called the 'technical-conceptual cut'[3]. Indeed, theoretical approaches used at that time in CAS research, according to the authors, were of a constructivist nature[4] but, in our opinion, tended to use this reference to constructivism in order to caution in some sense the technical-conceptual cut, and we felt the need to take some distance from these.

Anthropological and socio-cultural approaches seemed to us more sensitive to the role played by instruments in mathematical work and to be able to take proper account of the role of "technical work"[5]. This is the reason why we turned our attention towards the anthropological approach developed by Chevallard (Chevallard, 1992; Bosch & Chevallard, 1999), which has become very influential in French educational research. This approach, with its institutional basis, also allowed us to give proper place to institutional issues which, more and more, we recognised as essential. As it is obviously impossible to summarise in a few lines the anthropological approach, I will only point out the main elements necessary for understanding the following discussion.

2.1 The anthropological approach

The anthropological approach shares with 'socio-cultural' approaches in the educational field (Sierpinska & Lerman, 1996) the vision that mathematics is seen as the product of a human activity. Mathematical productions and thinking modes are thus seen as dependent on the social and cultural contexts where they develop. As a consequence, mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. The

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[2] CPGE classes are, in France, special classes where selected students prepare for the difficult competitions which allow entrance to the most famous engineering and business schools. This is a two-year program after senior high school.

[3] By this expression, we labelled a common epistemological position (at that time) that, in mathematical activity, opposed what was considered, on the one hand as conceptual activity and, on the other hand as technical activity. Technical activity was there understood in a narrow sense as something mechanical, deprived from intelligence. Traditional teaching practices were accused to focus only on this technical dimension (through the development of procedures and skills), while meaningful learning was associated to the conceptual dimension. As CAS technology could take in charge most of the taught techniques, a common hypothesis in CAS research was that the use of CAS could allow students to work directly at a conceptual level (for more details, see for instance the analysis of 175 publications around the educational use of CAS in the meta-study (Lagrange & al., 2000), and also (Lagrange, 2001)).

[4] As shown in (Lagrange & al., 2000), in most articles, constructivism had more or less the status of normal paradigm (Kuhn,) underlying educational research work. It was simply evoked through some few references, and some principles such as active learning, without more discussion.

[5] Technical work has here the wider sense it has in socio-cultural and anthropological approaches and is seen in a dialectic relationship with conceptualisation (see below, the presentation of the Chevallard's anthropological approach for more details).

word "institution" has to be understood in this theory in a very broad sense: family is an institution for instance. Any social or cultural practice takes place within an institution. Didactic institutions are those devoted to the intentional apprenticeship of specific contents of knowledge. As regards the objects of knowledge it takes in charge, any didactic institution develops specific practices, and this results in specific norms and visions as regards the meaning of knowing or understanding such or such object[6]. To analyse the life of a mathematical object in an institution, to understand the meaning in the institution of "knowing/understanding this object", one thus needs to identify and analyse the practices which bring it into play.

These practices, or 'praxeologies', as they are called in the Chevallard's approach, are described by four components: a type of task in which the object is embedded, the techniques used to solve this type of task, the 'technology', that is to say the discourse which is used in order to both explain and justify these techniques, and the 'theory' which provides a structural basis for the technological discourse itself and can be seen as a technology of the technology. Since I have already assigned a meaning to the word 'technology' in this article, to avoid misunderstanding, in the following I will combine the technological and theoretical components into a single 'theoretical' component. The word 'theoretical' has thus to be given a wider interpretation than is usual in the anthropological approach.

Note that, here, the term 'technique' has to be given a wider meaning than is usual in educational discourse. A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work. I would like to stress that techniques are most often perceived and evaluated in terms of *pragmatic value*, that is to say by focusing on their productive potential (efficiency, cost, and field of validity). But they have also an *epistemic value*, as they contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge. I will come back to this point later.

For obvious reasons of efficiency, the advance of knowledge in any institution requires the routinisation of some techniques. This routinisation is accompanied by a weakening of the associated theoretical discourse and by a "naturalisation" or "internalisation" of associated knowledge which tends to become transparent, to be considered as "natural"[7]**.** A technique which has become routine in an institution tends thus to become "de-mathematicised" for the members of that institution. This naturalisation process is important to be aware of, because through this process, techniques lose their mathematical "nobility" and become simple acts. Thus, in mathematical work, what is finally considered as mathematical is reduced to being the tip of the iceberg of actual mathematical activity, and this dramatic reduction strongly influences our vision of mathematics and mathematics learning and the values attached to these[8].
The anthropological approach opens up a complex world whose 'economy' obeys subtle laws that play an essential role in the actual production of mathematics knowledge as well as in the learning of mathematics. A traditional constructivist approach does not help us to perceive this complexity, much less to study it. Nevertheless, this study is essential because, as pointed out by Lagrange (2000), it is through practices where technical work plays a decisive

[6] One of the seminal works of Chevallard in this direction showed that "knowing fractions" does not have the same meaning for the French and the British middle school institutions. This results, at least partially, from the very different cultural relationships the two countries have with the metric system, which shapes the respective importance and role given to decimal numbers and rational numbers in the curriculum. More recently, B. Grugeon, in her doctoral thesis (Grugeon, 1995), showed that general and vocational high school develops very different institutional relationships to elementary algebra, and used this analysis in order to understand the failure of bright vocational high schools students entering general high schools, and to find ways of overcoming this failure.

[7] For instance, when students learn to solve first order equations, the different transformations they perform on equations (adding a number, multiplying by a non-zero number…) are carefully justified as transformations preserving equivalence. When solving such equations has become routine, this theoretical discourse tends to vanish, gestures such as passing one term from one member of the equation to the other one are naturalised and, for many students, tend to loose their mathematical meaning. At a more advanced level, all of us have routinised the passage from cartesian coordinates to polar coordinates in the computation of double integrals, but for many of us, this passage functions now as a natural gesture, and reconstructing the underlying theoretical discourse would be a non trivial task.

[8] We became more sensitive to this phenomenon when listening to the vehement discussions generated by the appearance of the Texas TI92 symbolic calculator in 1995, where the feeling expressed then by many teachers was that anybody, if given a TI92, could for instance succeed at the mathematics examination ending senior high school, because the TI92 performed all the necessary 'mathematical work'.

role that one constructs the mathematical objects and the connections between these that are part of conceptual understanding.

Technological evolution has upset this economy and the traditional equilibrium which existed between conceptual and technical work, and the dialectic interplay between the "ostensive" and "non-ostensive" objects[9] of mathematical activity (Chevallard & Bosch, 1999). The great reduction in the cost of execution that technology offers, for instance, reduces the need for routinisation work mentioned above. Techniques that are instrumented by computer technology are changed, and this changes both their pragmatic and epistemic values. The mathematical needs of the techniques change also: new needs emerge, linked to the computer implementation of mathematical knowledge and the representation systems involved (Balacheff, 1994). These needs are not easily identifiable if the mathematical activity is only attached to its "noble" part (the tip of the iceberg), and the mathematical needs of the technical work are not seriously taken into account. It seems to us that the anthropological approach furnishes an effective framework for questioning these changes and their possible effects on mathematics teaching and learning.

2.2 The Ergonomic Approach

The anthropological approach in didactics has not so far developed tools adequate enough for thinking about instrumentation processes, since it has developed with reference only traditional classroom environments. It was in the research field of cognitive ergonomics (which also adopts an anthropological perspective) that we found an approach for supporting our views about instrumentation (Vérillon & Rabardel, 1995). Researchers in this domain are used to working on professional learning processes which take place in technologically complex environments, for example the training of aeroplane pilots, and they have developed conceptual tools adapted to the study of such types of learning processes.

For us, the first contribution this approach makes is the conception of 'instrument' itself. The instrument is differentiated from the object, material or symbolic, on which it is based and for which is used the term "artefact". Thus an instrument is a mixed entity, part artefact, part cognitive schemes which make it an instrument. For a given individual, the artefact at the outset does not have an instrumental value. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes. Instrumental genesis works in two directions. Firstly, it is directed towards the artefact, loading it progressively with potentialities, and eventually transforming it for specific uses; this is called the *instrumentalisation* of the artefact. Secondly, instrumental genesis is directed towards the subject, leading to the development or appropriation of schemes of instrumented action which progressively take shape as techniques that permit an effective response to given tasks. The latter direction is properly called *instrumentation*. In order to understand and promote instrumental genesis for learners, it is necessary to identify the constraints induced by the instrument; and, especially for the type of instrument with which we are concerned here, there are two kinds of constraints: "command constraints" and "organisational constraints"[10]. These result from "internal" and "interface" constraints (Balacheff, 1994). It is also necessary, of course, to identify the new potentials offered by instrumented work.

2.3 One particular example: the case of "framing schemes"

Let us give one example. When students use function graphs in a computer environment (or a graphic calculator), they are faced with the fact that a function graph is "window-dependent" and they have to develop specific "framing schemes" in order to cope efficiently with this phenomenon. This is far from being a spontaneous and immediate process as many experiments have shown. For instance, in the research we developed with grade 11 science students (Artigue et al., 1998), in the first interview task students were asked to consider the function defined by $f(x) = x(x+7)+\frac{9}{x}$, use their TI92 to obtain an accurate representation of the function, make conjectures on its properties on the basis

[9] The anthropological approach emphasises the fact that mathematical objects are not directly accessible to our senses: they are "non-ostensive" objects; we work with them through ostensive representations which can be of very diverse nature: discourse in natural language, schemas, drawings, symbolic representations, gestures, manipulatives. Work with ostensive objects both shapes the development of the associated non-ostensive objects, and is shaped by the state of development of these.

[10] Command constraints are those generated by the commands available, their range of uses, etc. Organisational constraints are linked to the fact that working with a specific instrument influences the way someone plans and organises his/her mathematical work, taking into account its specific ergonomy and ways of functioning.

of this representation, and then test and prove these conjectures, and eventually explain why some of these were false. The function had been chosen with the following considerations:

- the function expression should be rather simple but the function itself of a type not familiar to the students (at that point, they were familiar with polynomial functions essentially);
- the graph obtained in the standard window, [-10,10]x[-10,10], should be far from being accurate (figure 1);

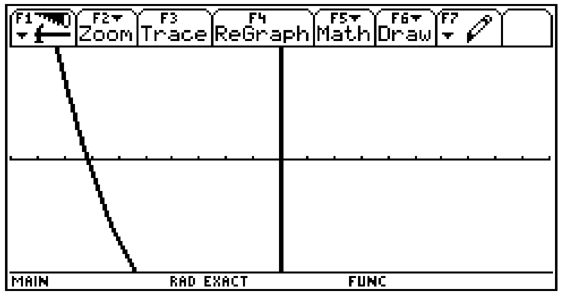

Figure 1: Graph in the standard window

- It should not be difficult technically to obtain an accurate graph, based on the different options offered by the calculator. For instance, one application of the command 'ZoomOut' is enough to obtain graph 2 (figure 2) and the command 'ZoomFit' gives graph 3. Students can also explore the values taken by the function through the 'Table' application and (without changing the default stepsize) by looking at the values of the function from x=-10 to x=10 can find an accurate window.
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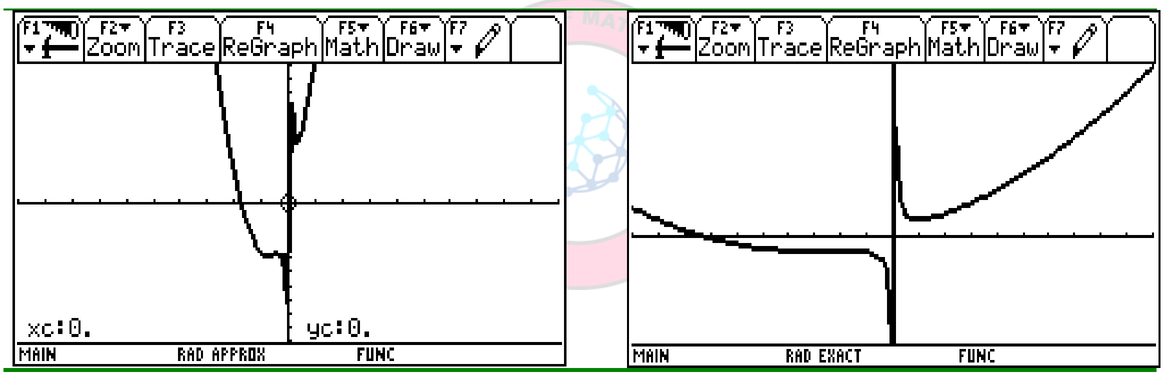

Figure 2: Graphs 2 (left) and 3 (right)

All these students had their own graphic calculator and had used it, in and out of class, for a year. So one could expect that they would have developed framing schemes allowing them to cope with this situation. What was observed? Among the nine students interviewed, chosen to reflect the different mathematics abilities of the students engaged in the experiment, as well as the different relationships they had developed with technology, only two succeeded, one using manual changes of the window, one through a manual change plus the use of ZoomOut and then ZoomBox (Defouad, 2000). Only two students only obtained an accurate graph for negative *x*, and the remaining five, in spite of many different manipulations, could not get something better than the graph in the standard window. No student used the Table application, or the HOME application (the calculator's main application for exact and approximate calculations) for exploring the values of the function, even though seven had already used HOME in order to define the function before asking for its graph. All of the students were clearly conscious that the graph drawn in the standard window was not accurate: a priori, they did not expect something monotonic, neither something which appeared just in the negative part of the x-axis, but they generally lacked any clear strategy to change the window, and gave up after a few trials.

I would like to stress that one should not see in this example the manifestation of some kind of cognitive inability. The results we observed are certainly due to the fact that, through the tasks these students had been introduced to graphing technology during their 10th grade mathematics courses on functions, they simply had not been faced with the necessity of developing such framing schemes.

In what follows, I shall try to show how the research we have carried out from the perspectives of the anthropological and instrumentation approaches has allowed us to progress in our reflections on the educational use of CAS, and I will focus on several specific points:

- The unexpected complexity of instrumental genesis,
- The mathematical needs of instrumentation,
- The status of instrumented techniques, the problems arising from their connection with paper/pencil techniques, and their institutional management.

These are not, of course, independent points even I separate them here in order to clarify the presentation.

3.0 THE UNEXPECTED COMPLEXITY OF INSTRUMENTAL GENESIS

During the last two years, as part of a national research project, we have carried out an extensive survey of technology in mathematics teaching (Lagrange et al, 2000, 2001). Its aim was to study the issues of the integration of technology into mathematics teaching and to better understand the difficulties of integration, and to review the potential and limitations of research and existing innovative work on TICE[11]. These difficulties are indeed persistent in France in spite of the continuous governmental support given to integration for more than 20 years now. The survey took into account more than 600 publications and reports, published between 1995 and 1998, of which 175 dealt with CAS. The results clearly show that the complexity of instrumental genesis has been widely under-estimated in research and innovation on TICE, until quite recently. The predominant role of pedagogical tool given to technology that I pointed out in the introduction has certainly contributed to such an under-estimation. Suggesting that instrumentation may be a complex and costly process does not fit visions that consider technology mainly as an easy tool for introducing students to mathematical contents and norms defined independently from it. In our research teams in Paris and Montpellier, two recent doctoral theses by Trouche (1997) and Defouad (2000), have allowed us to make some advances in the awareness and understanding of this complexity.

3.1 Trouche's thesis: student profiles and instrumented schemes

Trouche's thesis mainly concerns the conceptualisation of the notion of limit, in two different environments, graphic calculators and symbolic calculators. The author especially focuses his attention on limits in the neighbourhood of infinity. Students' conceptions and their evolution through an engineering design project which covers the teaching of calculus and elementary analysis for the whole of grade 12, in the 'scientific track' of the senior high school programme12 are investigated. Trouche's research firstly evidences the diversity of instrumental relationships that the students develop in the institutional context of the high school. This diversity led Trouche to introduce five extreme profiles, which he calls "theorist", "rationalist", "scholastic"[13], "tinkerer", and "experimentalist". These are characterised by the kind of resources favoured by the student, the meta-knowledge she tends to activate, and the modes of validation she privileges (Trouche, 2000). The "tinkerer" for instance is characterised along these three axes by the triplet (calculator, investigation, accumulation), while the "rationalist" is characterised by (paper/pencil, inference, proof) and the "theorist" by (references, interpretation, analogy).

According to their profile characteristics, students develop different relationships with their graphic and symbolic calculators. Trouche (2000) illustrates this point first by analysing students' behaviour when, rather early in the academic year, they are asked to study with their graphic calculator the limit in $+\infty$ of a polynomial function of degree 4 whose x^4 coefficient is 0.03. The small size of this coefficient makes the graphical representation of the function in the default calculator windows incoherent with the "algebraic" study these students are theoretically able to produce at that time of the academic year. Trouche shows that, for this task, students develop a wide range of solving strategies, associated with different use of the graphic calculator, and that these differences can be interpreted

[11] TICE: Technologies de l'Information et de la Communication appliquées à l'Enseignement.

[12] In the Baccalaureate system, high school students specialise into different tracks. General high school offers three tracks : L (literature, philosophy and languages), ES (economy and social sciences) and S (sciences). Tracking begins at grade 11.

[13] We translate by "scholastic" the French word «scolaire» which has the following meaning: a student is said to be "scolaire" if (s)he mainly tries to adapt to the institutional constraints and succeed through this adaptation, if (s)he mainly functions by taking into account the didactic contract. Guin & Trouche (1999) used the following English translations in their account of this research: "theoretical", "rational", "random", "mechanical", "resourceful". Note that these profiles have to be considered as prototypes, used for categorising and analysing observed students' behaviour, not the students themselves who, even if they look closer to one profile than to another one, cannot generally be reduced to such prototypes.

in terms of profile characteristics. Then, through regular observations of the same students (once a week they are proposed an open problem to solve in a two-hours session) he shows that differences in the students' profiles have significant effects on the instrumental genesis of graphic calculators (and also on the instrumental genesis of symbolic calculators which were introduced later in the experiment).

Trouche's research allows him to identify the different schemes of instrumented action which are likely to appear in work on limits at this level of schooling. These schemes rest on various "theorems in action" (according to the terminology introduced by Vergnaud in the theory of conceptual fields — Vergnaud, 1990), and dominant schemes evolved during the year for each type of student profile. For every profile except the "scholastic", instrumented action, initiated in the environment of the graphic calculator, does not tend to reduce to the mere use of the symbolic calculator command "limit" with its specific syntax; when symbolic caculators become avalaible: the schemes that the students developed are more subtle than that (and of course, the activities which are proposed to students and their didactic management certainly play an essential role).

The research also shows that the local schemes that develop for dealing with specific tasks, for example: the determination of limits in $+\propto$, interact with more global schemes, for instance the one called by Trouche "scheme of approximate detour". This scheme corresponds to the act of using a specific command available on the TI92 and TI89 calculators ("diamond + ENTER") to get an approximate value for a given number, whilst working in the calculator's "exact" mode. This scheme can be used in diverse situations where the calculator cannot give an exact answer[14] or when one wants to find out more about a result whose exact expression is not informative enough (at least as regards the magnitude of the number). But Trouche also finds that this scheme can take different meanings, according to the student and the situation: it can be part of a process of anticipation or verification (DA1); it can be considered as a substitute for intended result and used as such (DA2), it can be seen as a stratagem in which the peculiarities of the approximate value which is obtained are used for guessing the exact value (DA3). It seems that uses of type DA1 are more likely to appear with "rationalist", "theorist" and "experimentalist" profiles, DA2 with "tinkerer" and "scholastic", and DA3 with "tinkerer".

3.2 Defouad's thesis: The genesis of students' TI92 instrumentation for the study of function variations[15]

The thesis by Defouad is concerned with the instrumental genesis of the TI92 calculator, and focuses on just one task, which can be considered as emblematic of students' entry into the field of elementary analysis: the study of function variations. This research was situated in a larger experimental context which covered the entire mathematics curriculum for scientific tracks at grade 11, with the students in an experimental class being given a TI92 calculator for one academic year. Grade 11 is the point where, in France, elementary analysis and calculus begin to be officially taught. The methodology of the experiment combined regular classroom observations, questionnaires and tests, and the specific observation throughout the academic year of nine selected students in the experimental class (according to their sex, academic level in mathematics and their personal relationship to technology).

The task we consider here is no longer a local task and this fact, of course, influences the vision it gives of the instrumental genesis. This research, like the preceding, showed the complexity of instrumentation processes. Instrumental genesis develops over time in ways that do not reflect the temporal organisation of the formal teaching devoted to the topic of function variation. The genesis combined a succession of cycles ("explosion – reduction"), and even at the end of the academic year it remained fragile. There was also a tendency among the students towards a different relationship to mathematical proof, in that they gave more importance to the search for coherence between information coming from different sources (symbolic computations, graphical representations and the approximate values obtainable through the commands available within the graphical application, tables of numerical values) than to the search for a decisive argument.

[14] For instance, when the calculator is not able to give the exact solutions of an equation, to compute the exact value of a limit or a definite integral…

[15] The study of function variations is a fundamental type of task regarding functions in the French high school curriculum. For instance, students are proposed some geometrical object where some magnitudes (lengths, areas, volumes) depend on the choice of one variable magnitude (generally a length x), and asked to study how these magnitudes vary when x varies, to find extremum values… When they are asked to study the variations of a function, with or without any particular context, French students know that they have to find the intervals where the function is increasing (respectively decreasing), the local and global extreme values and, when they have been introduced to the notion of limit, to compute limits if necessary.

More precisely, by relying on the data collected in the regular interviews of the selected students[16], Defouad identified several phases in the instrumentation of variation. During the first phase, the students remained strongly attached to the culture of the study of functions that they already had been introduced to at grade 10, with graphic calculators. This culture is of a numerical-graphical-algebraic nature, the variation of a function is essentially inferred from the reading of its graphical representation and, to a lesser extent, from tables of numerical values. Adapting this culture to their new situation, in the first phase, the students considered the graphs of a function and its derivative as their main tools for conjecturing and justifying its variations. The use of formal calculus in the 'HOME' application of the symbolic calculator, in spite of its potential usefulness, remained marginal. HOME was nearly always used just for defining functions, and calculating or checking the value of their derivatives. In the classroom observed sessions, the pressure of the didactic contract made such strategies soon enough disappear, as an evident clause of the didactic contract is that students are supposed to use the new mathematical concepts and techniques[17] they have been introduced to, when asked to solve a particular task. In the interviews, the lower pressure of the didactic contract associated with the fact that students were faced with less familiar functional objects, provoked behaviours that evidenced the strength of this first enculturation.

Let us give a particular example: that of Frederic, a student with a rather good level in mathematics and a positive relationship with technology. At the first interview, about two months after the introduction of the derivative, he was asked to study the variations of the function: $f(x) = x(x+7)+9/x$ (cf. section II.3). He firstly defined the function in the HOME application, then went to the 'Y=' application and asked for the graph in the standard window. As we have seen, this strategy leads to a very partial picture and he was not satisfied, but, after one more trial, his interpretation was that the graph is included in the half plane corresponding to negative x. Without checking this interpretation by looking at the algebraic expression, he decided to reduce the window to negative x and then to adjust the vertical interval in order to make visible at least one extremum point[18]. This was done by trial-and-error, guided by some idea of reasonable form but without any connection with the algebraic expression of the function. The graph he finally obtained is reproduced in figure 4.

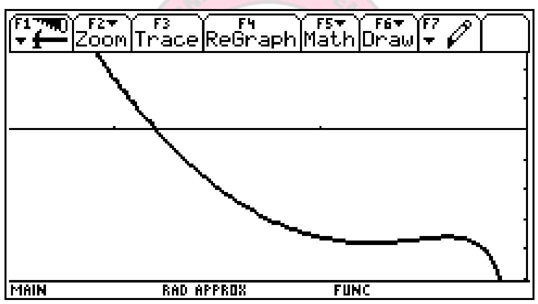

Figure 4: Frederic's final graph

Then Frederic jumped to HOME, asked for the derivative, and then its factorisation, but was apparently unable to make use of it. Clearly, he was puzzled by the complexity of the expression he obtained and did not understand that the factorisation could give him the sign of the derivative[19]. So he came back quickly to the graph application, graphed

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[16] During each interview students were asked to study the variations of a function given by its algeraic expression. One important characteristic is that this function was deliberately choosen beyond their area of familiarity, in order to analyse the ways they adapted to non routine situations.

[17] That is to say, in this particular context, use paper and pencil or the Home application in order to calculate the derivative, determine its zeros and sign, and rely on the 'variation theorem' in order to link the positive (resp. negative) sign of the derivative on intervals to the increasing (resp. decreasing) nature of the variations of the function on the same intervals.

[18] In his opinion, this expression was certainly too much complex for corresponding to a monotonic function.

[19] For the students at that stage of the course, the calculation of the derivative would normally result in a second degree polynomial, or possibly a fractional expression whose denominator is a square and numerator a first or second degree polynomial. For second degree polynomials, they knew the specific rule for the sign of P(x) according to the position of x with respect to the roots of the equation P(x)=0. They had been taught in their grade 10 algebra course how to deduce the sign of any algebraic expression from its factorised form in first degree expressions, but they had not internalised this technique as an operational tool in functional contexts because the set of available

the derivative and used jointly the information given by the two graphs, and the table application, in order to make conclusions about the variation of the function (figure 5). The function looked first decreasing, then rather flat, then decreasing, but the derivative was first negative, then positive, then negative. Thus he conjectured that the flat part was certainly in fact an increasing part, between a minimum and a maximum.

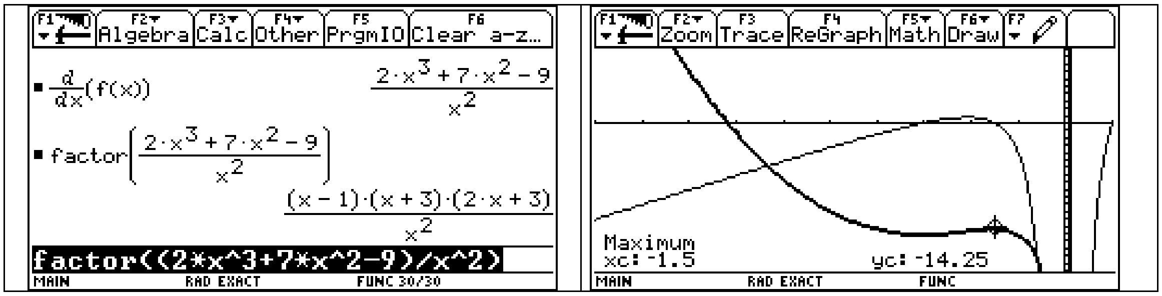

Figure 5 : Frederic studying the variations of the function

He then checked his conjecture by using the 'math menu' of the graph application for finding the extremum, then by using the Table application and, finally, by zooming on the apparently flat part of the graph (see figures 5 and 6). At that point, he looked satisfied but he had failed to notice the part of the graph of the derivative corresponding to positive x and needed the help of the interviewer to interpret it when his attention was drawn to it.

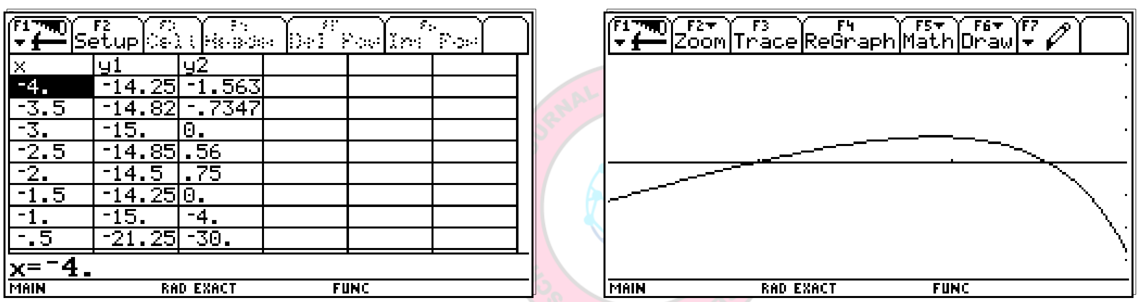

Figure 6 : Frederic checking his work on variations

At the second interview, two months later, there was an evident development but Frederic was still in what Defouad calls the "intermediate phase": the graphic application still plays an essential role but HOME is beginning to emerge as a real solving tool and the connections between the different applications are developing. Frederic had not mastered the graphical phenomena associated with vertical asymptotes, and obtained an evident discrepancy between the reading of variations from the graph of the function and the reading he makes from the graph of the derivative. Nevertheless, he was eventually able to solve this contradiction for himself.

At the specific assessment[20] we organised in June, he was clearly in what Defouad calls the "calculus phase", having developed specific and efficient instrumented schemes for framing and variation analysis, by connecting the symbolic and graphical applications of the calculator. But, soon after, at the third interview, he was faced with a new type of function, mixing square roots and trigonometric functions, generating new phenomena linked to the discretisation processes used by the calculator, and this perturbation was a bit too much for him. He spent a lot of time trying to produce a graph that exactly touched the x-axis (see figure 7), doubted about the periodicity of the function,

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functions and functional tasks was too narrow at that time. All this context contributes to explain Frederic's unexpected reaction.

[20] All the grade 11 classes of the scientific track, in the high school, had regular common assessments, and the students of the experimental class were allowed to use their calculator for these common assessments. But in June, we organised a specific assessment for the experimental class, with questions asking sudents to interpret data provided by the calculator, both in the Home application and in the Graphic application, to explain the errors made in a false reasoning using the calculator presented to them, and to solve a problem on functions they could not solve without using the calculator (Artigue 1 al., 1998).

and when he got the formal expression for the derivative, he was completely stuck. He asked for particular values of the derivative but needed help in order to prove the conjecture he had made about its sign from the graph.

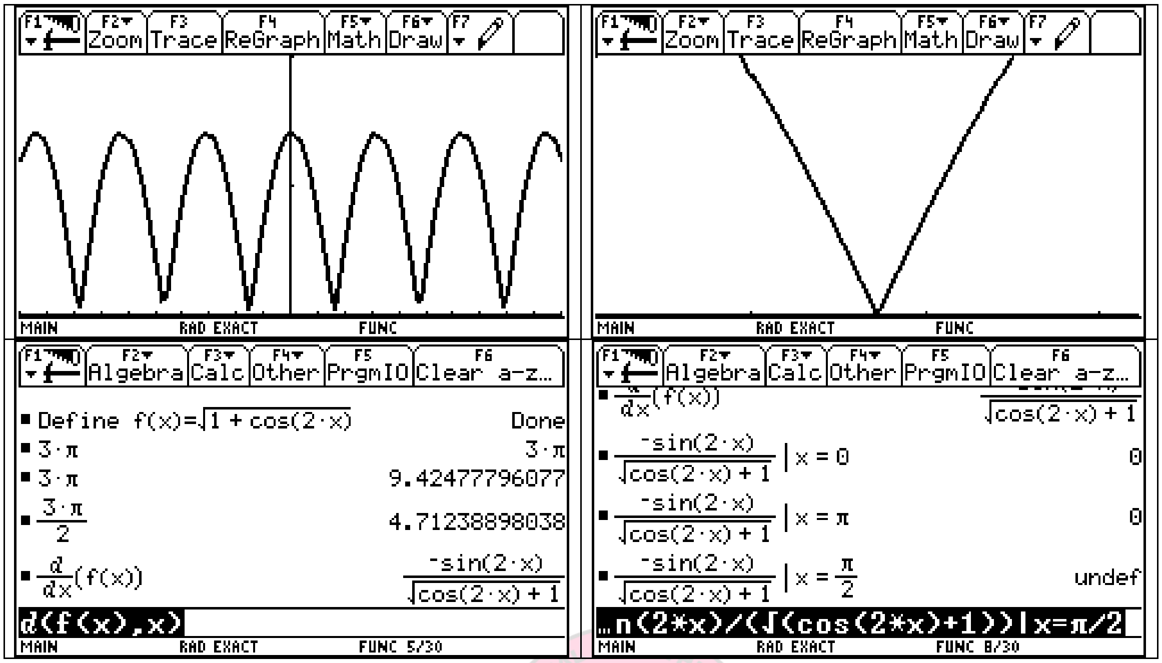

Figure 7: Excerpts from Frederic's work in the third interview

The case of Frederic is representative of all the students in the experiment. During the first year of the experiment, students came to use the HOME application for dealing with variation issues very slowly, with clear differences according to their personal profile. The development of their mathematical knowledge played an important role in this progressive appropriation. In the first phase, we noticed that, even if the students were able to use the HOME application for computing with the derivative (eg. finding its zeros), they came back to the graphical window as soon as the situation or the results provided by the calculator became more complex. For instance, they were not aware of the fact that, once an algebraic expression depending on one variable is factored, one is able to easily find its sign for any value of the variable, and that it does not depend on the number of factors. The economic strategies of use for the TI92 that we imagined before each interview were rarely the strategies the students chose. They preferred "zapping[21]" between applications and "over-verification" strategies. The efficient connection between the algebraic and graphic registers needed more time to develop than we expected.

Instrumental genesis involves an evolution in the roles of the different applications of the calculator. As I mentioned above, during the first phase of this genesis, the graphical application plays a predominant role both in exploration and solving; the Table application plays a control role; the symbolic application HOME plays a marginal role (computation of the derivative essentially). In a second phase, even if the graphical application still predominates, HOME becomes more involved in the computation of exact values for the function and its derivative, for calculating limits, or checking some graphical results, with a role of support to the graphical work. In the third phase: the so-called calculus phase, an inversion takes place: the symbolic application becomes the predominant tool in the solving process, jointly with paper/pencil work, while the graphical application becomes mainly an heuristic tool, for anticipation and control. During the first year of experimentation, our attention was attracted by the slowness and circuitousness of this instrumental genesis. This led us to question the status of instrumented techniques in the experimental class and the ways this status could have influenced the observed genesis. I would like to come now to this second aspect of instrumentation.

[21] This expression was chosen by analogy with a very common behaviour of TV viewers where one moves rapidly from one channel to one another.

4.0 INSTRUMENTED TECHNIQUES: A PROBLEMATIC STATUS, EVEN IN EXPERIMENTAL ENVIRONMENTS

In a CAS environment, teaching combines two types of techniques: paper/pencil techniques and instrumented techniques. Every technique has, as we pointed out above, a pragmatic and an epistemic value. The institutional status of techniques depends on the values attributed to them. It is certainly easy to recognise the pragmatic value of instrumented techniques, but it may be less easy to grasp their epistemic value. To a great extent, the problem lies in the immediateness of results, compared with our familiarity with the idea that the epistemic value of a paper/pencil technique becomes evident through the details of its technical gestures. Let us take a very simple example: the technique of Euclidian division. The epistemic value of this is evident: it plays a fundamental role in the proof of various arithmetic theorems, and it helps to explain their necessity and the connections between them. At a more elementary level, through the iteration of the division gesture, pupils can understand very early why the decimal expansion of a rational number is necessarily periodic. The use of a calculator, which gives the beginning of the decimal expansion of any rational number instantaneously, and in most simple cases allows the student to conjecture about both the periodicity and the actual period, no longer has the epistemic value of the paper/pencil gesture.

In general, the epistemic value of instrumented gestures is something that must be thought about and reconstructed. In the teaching process, it has to be developed through an adequate set of situations and tasks. The experiments that we have carried out, and also those undertaken by other researchers (see for example Schneider, 2000; Kendal & Stacey, 2001) show that this is not trivial, even when a detailed analysis of "spontaneous evolutions" of teachers working in CAS environments show that some of these can be interpreted *retrospectively* as attempts in that direction. I will come back to this point later on as it requires deeper analysis. What I simply want to stress here are the difficulties that, even in special experimental settings, teachers encounter in giving an adequate status to instrumented techniques and in managing these from an institutional viewpoint. The observations we made during the first year (as analysed by Defouad) illustrate this, by revealing the very different lives that paper/pencil and instrumented techniques had in the experimental class.

In a French classroom, today, work on a new mathematical theme begins with a first phase of exploratory work, after which certain paper / pencil techniques are identified and become "official". Students are then trained to use these techniques in different contexts and some routinisation work takes place. A discourse of a more theoretical nature goes along with this institutionalisation of official techniques, having explicative and justificative aims, even if crucial theorems (such as the theorem linking the sign of the derivative and the variation of the function (see note 18)) are not formally proved at this level of schooling. The institutional life of instrumented techniques, such as those called up in the study of the variation of functions (techniques for adequately framing graphical representations, for finding the sign of the derivative, for checking the equivalence of algebraic expressions, etc) is quite different.

The symbolic calculator leads to an explosion of possible techniques which can be used for solving each of these tasks: think for instance about the number of different zoom commands offered in the calculator's graphical application, the different schemes one can use in order to get the sign of an expression, or to check an algebraic equivalence. Faced with such a situation, teachers encounter difficulties and don't dare to make the necessary choices, which they certainly would make (often unconsciously) in a more traditional environment. There is a temptation to allow students to discover up to what point the calculator can enrich their solving strategies, and this can result in a non-productive technical overload. When the educational institution does not give teachers any rule for selection, they are less sensitive to the necessity of making choices, and not equipped to make these in a rational way. This leads to an explosion of techniques which remain relatively ad hoc, and pose a didactic obstacle to the progressive building of mathematical activity instrumented in an efficient way.

Another difference may be added to the preceding ones. Any technique, if it has to become more than a mechanically learnt gesture, requires some accompanying theoretical discourse (see for instance note 3). The kinds of discourse which can be developed are well-known for official paper/pencil techniques, and moreover these are framed by the syllabus, textbooks and other educational resources. A discourse has to be constructed for instrumented techniques. Once more, difficulties are obvious because this discourse will call up knowledge which goes beyond the standard mathematics culture. It will necessarily intertwine standard mathematics knowledge and knowledge about the artefact and the "computational transposition"[22] of mathematical knowledge that the use of this artefact involves.

[22] In the sense of Balacheff (1994): Referring globally to the concept of didactic transposition (Chevallard, 1985), Balacheff points out that the implementation of mathematical knowledge in computer software makes the transpositive process more complex by adding a new layer to it: the computational transposition. This computational transposition affects the mathematical functioning of knowledge through processes internal to the computer (the implementation of data and algorithms) but it also affects the way we access to knowledge through the

For instance, knowledge underlying the mastery of framing techniques in the graphical application includes mathematical knowledge about discretisation processes and their possible effects, and also more specific knowledge about the way the artefact implements these discretisation processes. Knowledge underlying the mastery of algebraic equivalence includes knowledge about the principles which govern the representation of algebraic and numerical expressions in CAS software, about problems such as the existence of canonical or normal forms (Kovacs,1999), and about the differentiation between semantic and syntactic equivalence. As stressed by Defouad, knowledge related to the peculiarities of the artefact is often difficult, if not impossible to access. This fact led him to refine the typologies of constraints previously introduced by Balacheff and Trouche, by taking into account four different levels of information accessibility, ranging from information immediately accessible at the interface of the artefact to information completely inaccessible to the user[23]. Moreover he shows that, very often, the theoretical discourse necessary for a reasonable control of instrumented techniques developed by the students requires information situated at least at the third level.

Under these conditions, it is reasonable to believe that building a theoretical discourse relevant to some given instrumented techniques and well adapted to the students' cognitive state is not a trivial task. The a posteriori analysis of the classroom observations made at the end of the first experimental year confirms this conjecture: the theoretical discourse developed about instrumented techniques had been rather poor, episodic and lacked a clear structure. It was not clearly incorporated into the institutionalisation process[24] which essentially dealt with knowledge relevant to mathematical work in the paper/pencil environment. These characteristics did not help instrumented techniques to gain mathematical status and tended to reduce their epistemic value. Even when fully legitimated, they remained at a sort of intermediate status in the classroom culture, for which Defouad introduced the notion of "locally official techniques".

We were not sensitive to all of these differences at the beginning. Our attention was first attracted by the "explosion-reduction" dynamics mentioned above, by the time necessary for reaching a reasonable state of stabilisation, and the great diversity of stable personal strategies. The analysis of the data collected during the classroom observations allowed us to identify the differences in status and management we have mentioned and to build the interpretation I have articulated above. These were incorporated into the engineering design for the second experimental year. The teacher, now sensitive to the difficulties, tried to face them, and, we helped her to develop the explanation, justification and institutionalisation for the instrumented techniques she chose to favour and give an official status to in the class. Also in the design, we paid more attention to the necessary evolution of the didactic contract about instrumented techniques, through the academic year, taking into account both the evolution of instrumentation and mathematical knowledge.

This strategy had evident results. For instance, in the first interview of the second year, many students were no longer trapped by the simplest graphic phenomena linked to the choice of windows or to discretisation. Moreover, to some degree they were able to articulate reasons for these phenomena which linked mathematical and instrumental knowledge. We also had the feeling that the gap observed between classroom behaviour and interview behaviour the year before had decreased and that, even if the global dynamics were the same, we observed in the first interview competencies that we only had observed in the second, during the year before. Nevertheless, the students' instrumental knowledge remained fragile as did the mathematical knowledge they begin to built in the field of analysis. If we compare what we achieved with, for instance, what was achieved by Trouche with his 12 grade students (Trouche et al, 1998) during the second year of his experimentation, there is an significant qualitative difference.

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characteristics of the interface. The first analysis of such computational transposition was developed by Balacheff about the software Cabri-géomètre.

[23] The four level are the following ones : level 1: the information is directly accessible at the interface (for instance the mode of computation : exact, automatic, approximate, is coded in the status line at the bottom of the screen) ; level 2: the information is not directly accessible at the interface but is easily accessible (for instance the fact that, even if the calculator is in exact mode, all the computations made outside the HOME application are approximate computations, or the kind of transformation corresponding to a command such ZoomIn) ; level 3: the information is accessible but not so easily. One cannot find necessarily it in the documentation given with the calculator, but has to go to specific documents or websites (steps used in numerical integration…) ; level 4: information non accessible to the user (the choices made in the coding of objects, in the implementation of algorithms…).

[24] The institutionalisation process is, according to the theory of didactic situations (Brousseau, 1996), the process by which knowledge built by students in the classroom in the solving of problems is linked to the institutional forms of knowledge that the teaching aims at. This process is a fundamental process under the responsibility of the teacher who is the warrant in the classroom of this institutional knowledge.

5.0 THE MATHEMATICAL NEEDS OF INSTRUMENTATION AND THE EPISTEMIC VALUE OF INSTRUMENTED TECHNIQUES

I introduced earlier a distinction between the pragmatic and epistemic values of techniques. I also pointed out the mathematical needs required for an efficient instrumentation. These include some things that do not have a significant role in traditional mathematics teaching, designed for learning in standard environments. In order to make this point clear, I will come back to two questions already considered: the question of window framing and the question of algebraic equivalence.

5.1. Framing, discretisation processes and associated mathematical needs

Mastering the techniques of graphical representation associated with the study of real functions, in a paper/pencil environment, does not oblige learners to understand the possible effects of discretisation processes in dealing with computer representations of functions. In order to check this assertion, it is enough to propose to advanced students in mathematics, or to teachers unfamiliar with computer graphics, situations such as those which have been developed at the Montpellier IREM[25] (Trouche, 1994). For instance the following: students are given different graphical representations of the real function f defined by $f(x)= \sin x/x$ for positive x and $f(0)=1$. Some representations have a lot of oscillations of decreasing amplitude (as expected, but the number of oscillations can be very different), two are strictly monotonic (either increasing or decreasing), with the x-axis as an horizontal asymptote, and one is horizontal on the x-axis. Figure 8 shows two of these representations

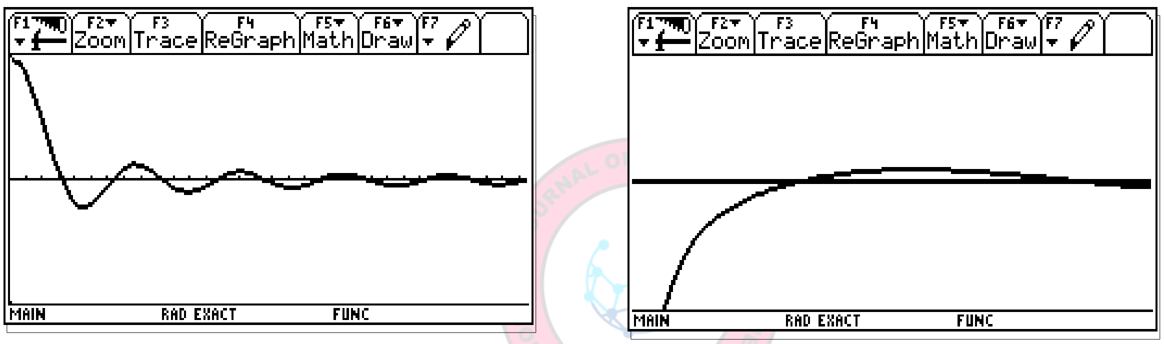

Figure 8 : Two graphs associated with the function sin *x*/*x*

As shown in figure 9, zooming on a monotonic part of the graph can result in a graph with very strong oscillations.

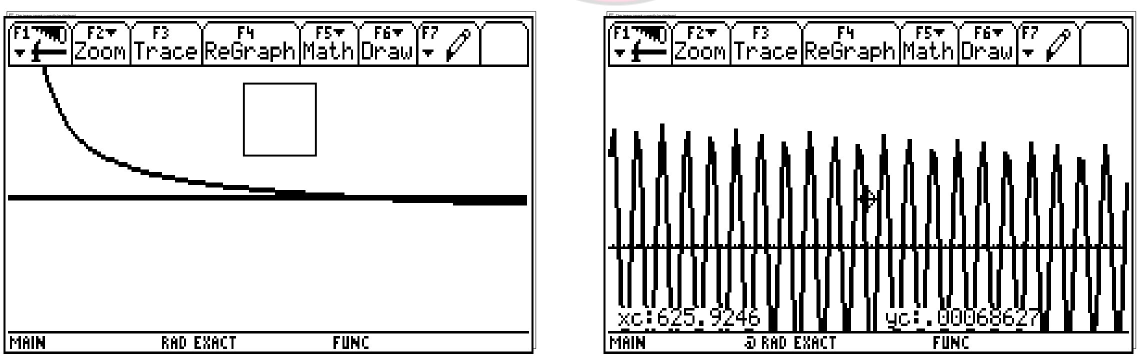

Figure 9 : Zoombox applied to a monotonic graph

Students were asked to explain why so many different graphs are possible and to find ways to reproduce analogous behaviour on their personal calculator, and more generally on any kind of graphic calculator[26]. This was

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[25] IREM: Institute of Research in Mathematics Education.

[26] In order to reproduce the same phenomena on a given graphic calculator, one has to know the number of pixels of the screen for this calculator and adapt the window to these characteristics. For instance, choosing the window so that pixels correspond to numbers close to $\pi/2+2k\pi$, gives a decreasing curve.

not a trivial task at all for them, since mastering it requires a clear understanding of the properties of the function, of discretisation processes, and of the particular characteristics of the machine they are using.

5.2. Equivalence between expressions

The question of equivalence between expressions is a more complex question which is linked to crucial theoretical problems. As stressed by Kovacs (1999) in a recent survey on computer algebra: *In order to represent objects in a computer algebra system, one has to make choices on both the form and the data structure level. The problem of "how to represent an object?" becomes even more difficult when one notices that some criteria (memory space, computation time readibility) may also play an important role in the representation. (…) It is now clear that it can be very hard to find a good strategy to represent objects in a unique way satisfying several criteria. Therefore, in practice, one object may have several different representations. This, however, gives rise to the problem of detecting equality when different representations of the same object are encountered. This in turn leads to the notions of canonical and normal representations and to simplification.* As Kovacs points out, many objects manipulated in formal calculus do not have a canonical representation, nor even a normal one. Equalities which are very easy to prove (by hand) such as $\sqrt{4+2\sqrt{3}} = \sqrt{3}+1$, are difficult to detect formally in systems working with algebraic numbers represented through polynomials[27]. Indeed, the solution of problems arising from the simultaneous manipulation of several algebraic numbers is at the core of current research in formal calculus.

Even when the context is more elementary, the use of a CAS obliges the students to face equivalence and simplification issues (all the experiments agree on this point). On the one hand, when entering an expression into (say) a TI92, the student is faced with the result of an evaluation automatically performed by the calculator. This result can look very different from the initial expression, and the student is in quite a different situation from paper/pencil work where, simplifying step by step, she knows the different intermediate expressions she has produced. On the other hand, algorithms implemented in the software lead to a greater diversity in the representations of mathematical objects than is usual in the classroom work with paper/pencil, where the representational forms are carefully chosen and the transformations between them are carefully codified, this codification remaining partly implicit as part of the didactic contract. CAS breaks these institutional norms by frequently producing much unexpected results. Equivalence problems arise which go far beyond what is usual for the classroom.

In the different experiments which we have carried out at high school level, we have paid specific attention to this point and used it as a lever to promote a work on the syntax of algebraic expressions, which is something very difficult to motivate in standard environments, where students often see it only as a matter of didactic contract; but in CAS environments, it appears as something necessary for efficient communication with the machine (for more details, see (Guin &Delgoulet, 1997), (Artigue, 2002). It was also a way of making the students sensitive to the fact that, even

if the CAS is a very powerful tool, some work remains the responsibility of the students, for instance because the software does not give the range of validity of the transformations it uses, as clearly appears on the last line of the screen in figure 10[28].

[27] Note: The TI92 automatically transforms the first expression given above into the second one and thus recognises the equivalence, but it does not recognise for instance the equivalence between $\sqrt{1/(2+\sqrt{3})^2}$ and 2-√3.

[28] If students for instance are asked to study a rational function given by the first expression, arising from some particular context, they cannot forget that the origin is a non regular point which requires some specific study. This is no longer visible on the simplified expression. The same occured for instance in the experimentation when using the TI92 for solving equations with radicals (Artigue & al., 1998).

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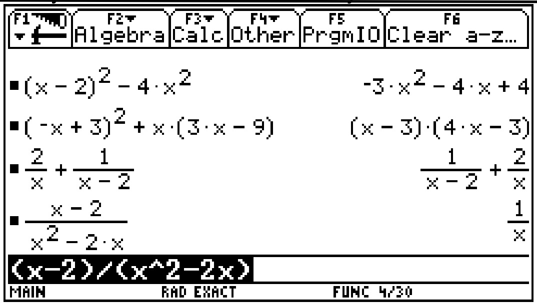

Figure 10 : Automatic simplification of an algebraic expression in the TI92

The mathematical needs we have just referred to are, at secondary level, needs linked with instrumented practices. One should add to the set of mathematical needs an understanding of the distinction between exact and approximate computations, as research carried out with CAS at secondary level shows the importance of this for the controlled use of these environments. Biribent has shown in his recent doctoral thesis (Biribent, 2001) that, for more than fifty years, French secondary teaching has been unable to cope adequately with these two facets of mathematical computations and that the introduction of digital technologies through scientific and graphic calculators has not improved the situation. The educational use of these technologies has not questioned the abusive hierarchy existing in the secondary educational culture between exact and approximate computations: approximate computation being what one does when exact computation is out of range, and having less mathematical value (for more details, see also (Artigue, 2002)). With CAS, the situation becomes rather different as the same instrument allows the students to perform both exact and approximate computations[29], and in our experiment, we have used this opportunity as a lever to develop a fruitful mathematical work about the distinction between these two forms of computations and their respective roles, and thus develop new relationships between these. We have only evoked here some of the mathematical needs that secondary students necessarily meet when working with CAS. At more advanced levels, new needs will necessarily emerge, especially those linked to the complexity and effectiveness of computations.

5.3. Coming back to the epistemic value of instrumented techniques

Up to now, I have expressed myself in terms of mathematical needs, but a change in viewpoint will allow us to come back to the question of the epistemic value of instrumented techniques, and through that to the question of the relationships between technical work and conceptualisation which is at the core of our reflection. Indeed, in our research projects, we first identified what we had perceived as the mathematical needs for efficient and controlled instrumented work. These needs went beyond the ambitions of current teaching and we asked ourselves how they could be negotiated and fulfilled, in an educational culture whose values were defined essentially independently of technology. We were thus especially sensitive to the conflicts between values that this situation could generate. A change in viewpoint was necessary in order to analyse these needs in terms of the epistemic value of instrumented techniques. Earlier, I pointed out that the epistemic value of instrumented techniques — that is, the way they contribute to the understanding of the objects they involve — was not immediately accessible to students. Looking back with hindsight on our research work, I have the feeling that the situations we have elaborated, tested, and progressively refined through the different experiments, and the results that they have allowed us to obtain, are very valuable in order to think about this issue.

Our experience and reflection leads me to identify two complementary sources from which mathematical work instrumented by CAS can aid conceptualisation and, more generally, to the progression of mathematical knowledge:

- the first source lies in the mastering of instrumented techniques, which at the beginning we considered rather negatively, as a constraint added by the environment;
- a second source lies in the new possibilities offered by instrumented work; this is a source easier to identify and, for that reason, more recognised in the literature about CAS.

The example of algebraic equivalence and simplification already mentioned is for me paradigmatic of the first point. For the grade 9 to 12 students in the experiments, it appeared as a useful tool for building didactic situations that

[29] More than that, scientific and graphic calculators can reinforce the existing dichotomy between exact and approximate computation by adding it an instrumental dichotomy: exact computations being performed by hand and approximate computations with the calculator.

allowed teachers to work with their students on issues linked to the exact-approximate duality, the syntax of algebraic expressions, the relations between semantics and syntax, and the status of numbers. Issues like this are difficult to address properly in standard environments, including those where graphical calculators, rather than CAS, are used (Artigue et al, 1995), (Guin & Delgoulet, 1997). The situations that we developed for that purpose do not have an immediate counterparts in paper/pencil or graphic calculator environments. They involve, generally, simple constructions and the experiments proved their effectiveness and easy management by teachers. Through these, we were able to partially compensate for the reduction in epistemic value resulting from the immediateness of the results in a CAS environment, mentioned above.

The example of access to generalisation through symbolic computation integrating the use of parameters[30] seems to me a paradigmatic example of the second source mentioned above, at secondary level at least. Secondary teaching in France works with particular mathematics objects (equations, functions…). Situations including parameters can be seen as a first step towards generalisation, accessible at a time when working with objects defined by generic conditions does not make sense. But these situations have progressively disappeared from teaching and textbooks in France, because algebraic computations involving parameters are considered to be more and more out of the range of students in the mass education which now predominates. Graphic calculators have allowed the introduction of families of functional objects, but the mathematical work involved tends to be situated at a graphical level. CAS allows the connection of such graphical work with symbolic work, so that regularities graphically observed can be proved in a symbolic way. This lever has been used throughout the ‘didactic engineering’ designs we have built for grades 11 and 12 (Lagrange, 2000), (Trouche, 1996). Its role has also been stressed in research work carried out at the Freudenthal Institute (Drijvers, 2000), and it is interesting to notice such a convergence in approaches between independent research projects.

We are clearly here in a different register: what counts is the potential that CAS offer for obtaining very quickly results, for reconsidering a previous computation and substitute a parameter to a numerical value in it, and the help CAS can offer as assistants to computation and symbolic proofs for students with limited technical background. Understanding the potential of CAS for learning and teaching mathematics requires, in my opinion, a deep reflection on the possible epistemic value of instrumented techniques, taking into account the two facets we have just mentioned. Epistemic value, of course, is not something that can be defined in an absolute way; it depends on contexts, both cognitive and institutional. From the contextual analysis of this potential to its effective realisation there is a long way, with situations to build, viability tests, taking into account the connection and competition between paper/pencil and instrumented techniques. And not forgetting the institutional negotiation of the specific mathematical needs required by instrumentation, a negotiation which, today, is not an easy one.

6.0 CONCLUSION

In this text, by looking back at different research projects, I have tried to tell a story of a developing consciousness and understanding for the researchers involved. It concerns the complexity of relationships between technical work and conceptualisation, and the crucial role of instrumentation issues. It also concerns the fact that these problems cannot be properly addressed without taking into account the institutional contexts, the constraints that institutions impose, but at the same time the potential they offer to mathematics teaching and learning, especially through the norms and values they define. Today, we have the feeling that we are able to set up our research questions in more adequate terms, and we have also the feeling that the research carried out up to now, allows us to better understand: the difficulties of the effective integration of CAS into mathematics teaching; some of the possible reasons for the success of some of our experiments and the failure of others; the ways in which the understandings we have developed could be transmitted to others. But, we are certainly very far from having definitive answers to the multiplicity of questions which have arisen from these research projects.

[30] Two examples of this generalisation process are analysed in (Artigue, 2002) in very details. The first one, for instance, is an optimisation problem in a geometrical context. The numerical value of the result obtained for the initial case suggest a relationship with some particular data and independence from the other ones. Introducing a parameter for this particular data instead of the initial numerical value allows the students to test this conjecture and solve thus a more general problem than the initial one.

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