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Discussing Basis Thinking around Arithmetic: the Principal stage of Mathematics

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Abstract
Mathematics is virtually a product of pure thought. Even though it draws its starting point from the world of experience, it refines them into concepts well beyond the scope of that world, and then, by purely logical processes of combinations, inference and constructions, builds up the most elaborate thought schemes. We now discuss the genesis of these thought schemes and how they were deployed. The set of numbers {1, 2, 3, 4…} are called counting numbers or, as mathematicians say, natural numbers. You will observe that zero is not a number of this set. However, in recent times zero has been added to this set as mathematicians thought that there is something very natural about zero. Now the set of counting numbers or Naturals look like this: {1, 2, 3, 4, …}
Keywords: *Dissecting Foundation, Mathematics, Organon Mathematics*

1.0 INTRODCUTION

Observe that these numbers are whole numbers are referred to as integers in mathematical language. Number theorists have repeatedly concluded that this set of numbers "are the most baffling stimulating and entertaining of all mathematical subjects." (Bergamini, 1971). When the operations of +, -, x, ÷, √ are applied, these numbers behave in particular ways that give rise to theorems that are used as models to describe and explain the phenomenology of the world. On the other hand, some observable phenomena of the world can be quantified by the use of integers and the application of theorems to these quantification gives new meanings and interpretations to these phenomena. The power of these interpretations lies in the possibility of making accurate predictions about the phenomena themselves. It must be pointed out that some works on the integers delve deep into the realm of abstraction from which there is no model to describe and interpret and physical world, but the models that emerge from this abstraction describe and interpret the transcendental nature of the set of Naturals themselves. Let us look at some elementary cases and see how mathematicians think and discover truths about the Natural numbers. Let us use the number line to show graphically the Natural numbers.

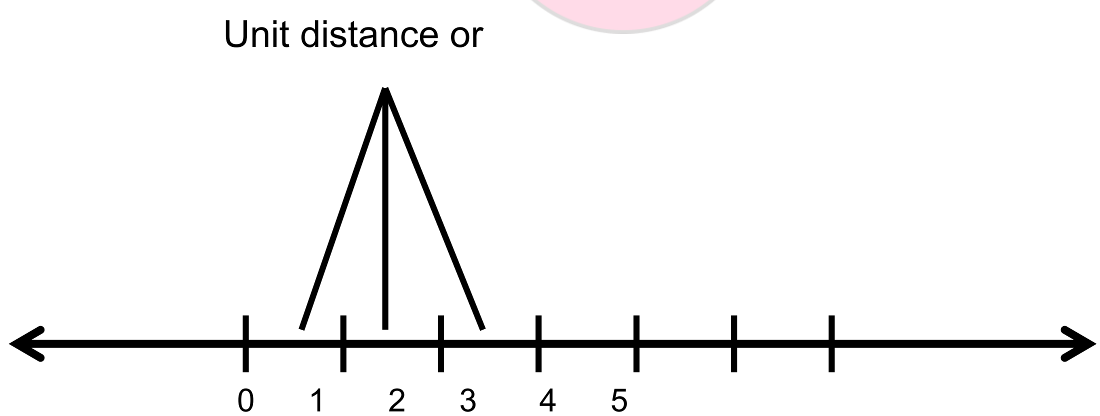

Figure 1: The Number Line

The mathematician looks at these numbers and observes that:

1. each number occupies a specific position on the line;
2. each number tells him how far it is from 0 in terms of units distance or length;
3. these numbers form a sequence , each increasing by 1 from 0 to the right;
4. if 1 is added to a given integer (number) the result is a natural number

Now if 1 is subtracted from 0 (which is now considered a natural number), the result is not a natural number. This is one exception to the rule and this limits the generalization of the rule to generate naturals by subtraction.

The mathematician now uses observation (4) and tries it out with several arbitrarily selected naturals. If 7 is chosen, he gets 7 + 1 = 8, which is a natural number. If 19 is selected he gets 19 + 1 = 20, a natural. If 72 is chosen, the gets 72 + 1 = 73, a natural. This process is called inductions, and the mathematician is said to be using inductive thinking; i.e. he examines several examples, searches for a pattern and then writes a general statement about that pattern. He concludes that if n is any natural number, then $n + 1$ is also natural. This is a simple and neat hypothesis. This is further tested by substituting naturals for n in (n+1). If this generates naturals, then (n+1) = a natural number and the theorem is established. If n is natural, then (n+1) is also natural. Testing of the hypothesis is an example of deductive thinking. But it is assumed that n is natural and that 1 by definition is natural. Therefore (n+1) is larger than the assumed largest. Therefore there cannot be a largest natural number. The naturals have no upper limit. (Intuitive).

Given that n is any natural number, then the next is (n+1), and the next is (n+1), and the next [(n+1) +1] and so on. The mathematician chooses, say, n, to be the first term in a sequence of naturals then he has:

1^{st} term n	=	n
2^{nd} term n+1	=	n + 1
3^{rd} term (n+1)+1	=	n + 2
4^{th} term [(n+1)+1]	=	n + 3
:		
K^{th} term	=	n+(k-1) …

So the general term of a sequence of natural number, starting with n is n + (k-1), where k refers to the k^{th} term. Substituting any natural values for n and k will always generate a natural number. Here the predicate of naturals is already subsumed in n and k, hence n+(k+1) will always be a natural number. This is intuitive reasoning. Now look again at the number line (Fig. 1). The mathematician observes that there are naturals that are divisible by 2, and that there are those that are not divisible by 2. He calls the set of numbers that are divisible by 2 even and the others odd. He observes that any given even number lies between two odd numbers, and that an odd number lies between two even numbers. He follows the even numbers, say from 8 reading left. He will get 8, 6, 4, 2. Now the next place holder for an even is zero. Is 0 an even number? Is it between two odd numbers? This is indeed a paradox. So the mathematician redefines zero. He says that 0 is the lower limit or just limit of natural numbers, and is, therefore, not subjected to the even number test, i.e. Even number = 2n where n is any natural number besides zero. But if n=0, then from above Even number = 2 x 0 = 0. But all even numbers are natural, and so is zero. The position that 0 is not subjected to the even number test is quite arbitrary and is designed to deliberately avoid including 0 as an even number. Maybe 0 is the first even number! Paradoxes lead to mathematical dilemmas. It is from dilemmas like this mathematicians create new mathematics. The mathematician looks at even numbers and studies them. {0, 2, 4, 6, 8, 10, …}. He observes that they are multiples of 2 and therefore divisible by 2. The above numbers are dissected as multiples of 2 like this {2 x 0, 2 x 1, 2 x 2, 2 x 3, 2 x 4, 2 x 5…}. Rewrite this as 2(0, 1, 2, 3, 4, 5, …). But the numbers within the brackets are the Natural numbers. Then the mathematician says that to generate even numbers multiply the natural numbers by 2. It n is any natural number, then 2n is natural and even. If 1 is added to the even number 2n, then 2n + 1 is odd and so is 2n-1. If n=0, then (2n-1) = 2 x 0-1= -1. This is not a natural number, so 2n-1 is not a general formula to generate odd numbers. So the mathematician settles for (2n+1). This will always generate odd numbers for all natural values of n. The mathematician asks questions like these:

1. Would the sum of even numbers be even?
2. What would the sum of an odd number and an even number be?
3. Would the sum of odd numbers be odd?
4. Would the square of an even number be even?
5. Would the square of an odd number be odd?

The mathematician may choose to select even numbers or odd numbers and apply the operation + to see what results would be so he could address the above questions. But this may not satisfy him; he needs general truths, so he uses what has already been established (by induction) to be universally true. An even number is the form of 2n and an odd number, 2n +1, where n is a natural number.

Question 1

Would the sum of even numbers be even?

Let 2m and 2n be two even numbers, where m and n are naturals.

Let 2m + 2n = 2 (m + n). Now 2 (m + n) is a multiple of 2, hence 2 (m + n) is even.
Look at the sum of 2m + 2n + 2p…2 z=2(m + n + p…z). But this is a multiple of 2 as m, n, p, …z are all naturals. So the mathematician concludes that the sum of any number of even numbers is an even number. This is an established truth and is now a theorem. This is deductive reasoning.

Question 2
What would the sum of an odd number and an even number be?
Let m and n be any two natural numbers then 2m and 2n + 1 are even and odd numbers respectively. Their sum is 2m+(2n+1) = 2m+2n+ 1. Now 2m + 2n + 1 can be written as 2(m+n) + 1. But 2 (m = n) is even. Then 2 (m + n) must be odd. The mathematician concludes that the sum of an even number and an odd number is odd. And again this is an instance of deductive reasoning.

Question 3
Would the sum of odd numbers be odd?
Let m, n, p, q, r… be natural numbers. Then
(2m + 1), (2n + 1), (2p + 1) etc. are all odd numbers.
Now (2m + 1) + (2n + 1) = 2m + 2n + 2 (m + n + 1). This is even.

Again (2m + 1) + (2n + 1) + (2p + 1) = 2m + 2n + 2p + 3 (m + n + p) + 3. This is odd. The mathematician may try a few more examples and soon discovers that the sum of an even number of odd numbers is even and that the sum of an odd number of odd numbers is odd. Here is an example of deductive reasoning.

Question 4
Would the square of an even number be even?
Let 2n be an even number. Then
$(2n)^2 = 4n^2$. This is even. Now an even number raised to any power, say power p, p being a natural number, we have $(2n)^p$. This is clearly an even number. The mathematician concludes that an even number raised to any power that is a natural number is even. Deductive reasoning.

Question 5
Would the square of an odd number be odd?
An odd number is in the form of 2n + 1.
Now $(2n + 1)^2 = 4n^2 + 4n + 1 = 4n (n+1) + 1$. This is odd. So the square of an odd number is odd.
The mathematician looks at this situation $(2n+1)^3 = 8n^3 + 12n^2 + 6n + 1 = 2n(4n^2 + 6n +3) + 1$. Now $2n (4n^2 + 6n + 3)$ is even. Therefore the mathematician concludes that $2n (4n^2 + 6n + 3) + 1$ is odd. Is this true for any power, say p, a natural number i.e. $(2n + 1)^p$?

From the above examples the mathematician observes that the sum of all the terms preceding the last term, which is one (1), is even. Therefore on adding 1, the last term in the expansion, to this even number makes the expansion odd. From this observation, the mathematician deduces that an odd number raised to any power, the power being a natural number, results in an odd number. This again is an example of inductive –deductive reasoning.

2.0 ON THE QUESTION OF AMBIGUITY

(1) Here is a classic example. It is agreed that division by zero is not defined, i.e. $x/0$ is not defined. What this is saying is that we cannot divide a number by zero. But why? Zero is a number and a natural one too! The mathematician say that $x/0$ does not exist, and we accept it. Look at this example $x/x-1$. This is a function of x, written in equation form as this

f(x) = $x/x-1$ reduces to:

f(1) = $1/1-1$ = $1/0$. This is not defined and therefore the function f(x) = $x/_{x-1}$ does not exist when x = 1.

This settles the question of division by zero, and for centuries this remained inviolable!

We now digress a little from the above and go back in time to the early Greeks who ushered in a new thinking which gave rise to a method of finding the area of plane figures bonded by curved lines.

The question of finding the area of a circle remained unanswered for a long time. The Greeks had already established the value of the ratio $circumference/_{diameter}$ as approximately equal to $3\,1/_7$. This ratio they referred to as π.

The formula for the area of a triangle was also established - $bh/_2$, where b is the base and h is the perpendicular height of the given triangle. Now the big question: how to find the area of a circle. Let us look at a new way of thinking by some early Greek mathematicians.

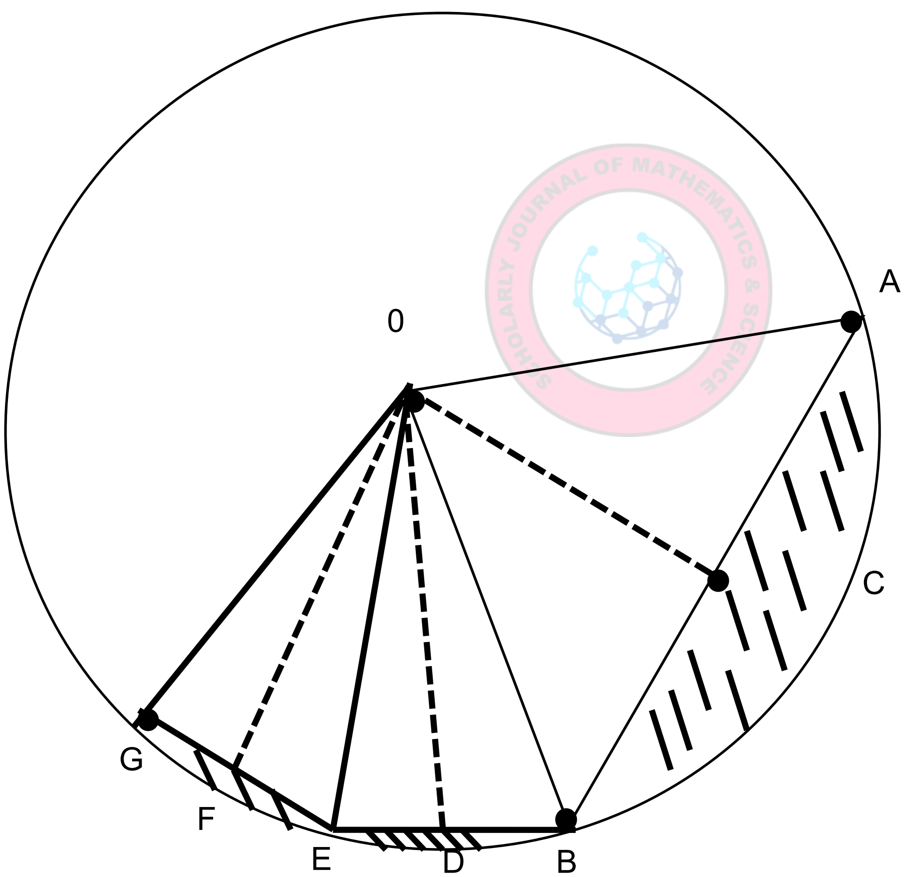

Fig. 2: Circle divided into sectors

If a circle is divided into sectors like OACB, then the sum of the areas of all such sectors is the area of the circle. Now the area of the triangle OAB triangle in the sector OACB can be easily found. The problem is how to find the area of the segment ABC.

Describe a smaller sector OBDE than the sector OACB. We see that in the sector OBDE, the height of the triangle OBE is greater than the height of the triangle OAB and that the segment BDE is smaller than the segment ACB. The sector OEG is smaller than the first two sectors and we observe the height of the triangle OEG is almost the length of the radius of the circle and that the segment EFG is very small. As the sectors get smaller and smaller, the segments they describe also become smaller and smaller and the heights of the triangles within the sectors approach the length of the radius of the circle and also area of the segments described by the sectors get smaller and smaller and the curvature of the segments approaches a straight line. This is a fortunate phenomenon as the size of the sectors become infinitely small, the sectors approach the shape of triangles whose bases form part of the circumference of het circle and their heights, the radius of the circle.

If there are n such sectors, when n is an 'infinitely large' number, then the size of each sector is infinitely small and approaches zero as n approaches infinity.

Let T denote an infinitely small sector which approaches a triangle in shape. Then $\{T_1, T_2, T_3 \ldots T_n)$ is the set of triangles into which the circle is divided. The area $T_1 = {}^{b_2 r}/_2$, $T_3 = {}^{b_3 r}/_2$ etc.

Sum of areas of all such triangles is the area of the circle and is given by:

$$A = \{ {}^{b_1 r}/_2 + {}^{b_2 r}/_2 + {}^{b_3 r}/_2 + {}^{b_4 r}/_2 + \ldots {}^{b_n r}/_2 \}$$

$$= {}^{r}/_2 \{ {}^{b}1 + {}^{b}2 + {}^{b}3 + {}^{b}4 + \ldots {}^{bn} \}$$ (sum of bases of triangle).

But the sum of the bases of all the triangles is the circumference of the circle, and this is known to be πd or $2\pi r$.

Thus, Area of circle = ${}^{r}/_{2} \times 2\pi r = \pi r^2$. And this formula is known to work.

However, the Eleatic school of philosophers strongly opposed the logic of slicing a circle into sectors and then considering a small sector as a triangle. Moreso as each sector becomes infinitely small or narrow it is no longer something but nothing. How can an infinite number of nothing become something such as a circle? Yet the idea of something becoming infinitely small so as to approach nothing gives a new meaning to zero. In this idea lies the gem of the infinitesimal calculus. Zeno, the leading spokesman for the Eleatics, was interested in eth idea of infinity. It is probable that his contemporaries thought that the universe was perfect and that all the things in it could be counted and expressed in terms of finite numbers. They thought that infinity was a bad idea, if not profane, and that it did not exist! Zeno disagreed. He confounded his fellow thinkers by this problem. Achilles can run ten times as fast as a tortoise. Achilles gives the tortoise a head start of 1 unit distance. Can he overtake the tortoise?

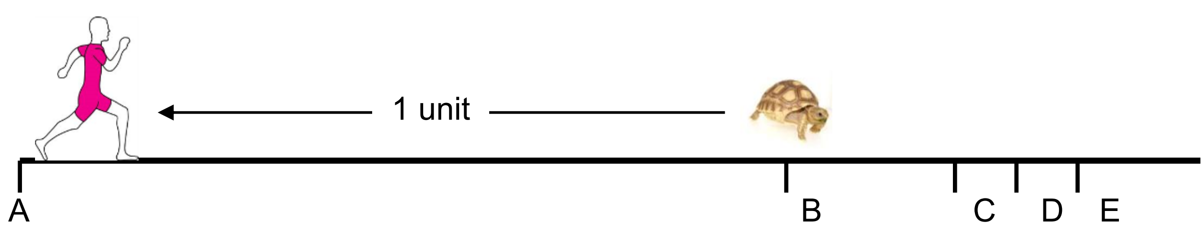

Fig. 3: Achilles and the Tortoise

Of course Achilles will catch up with the reptile, but how does one prove it by logical steps? This floored the Greek thinkers who sidestepped the problem. Here is Zeno's argument. If Achilles and the tortoise start at the same time from the positions A and B, respectively by the time Achilles reaches B, the tortoise would have moved ahead to C, where BC is $\frac{1}{10}$ unit, and by the time Achilles reaches C the tortoise would have reached D, $\frac{1}{10}$ of CD and so on. Whenever Achilles reaches a position where the tortoise was, it would be moved ahead. Will Achilles ever catch up with the tortoise? This is a paradox. From this emerges a new way of thinking. This paradox, when mathematized looks like this:

AB = 1 unit

BC = $\frac{1}{10}$ unit

CD = $\frac{1}{10}$ of BC = $\frac{1}{10}$ of $\frac{1}{10}$ unit = $\frac{1}{100}$ unit

DE = $\frac{1}{10}$ of CD = $\frac{1}{10}$ of $\frac{1}{100}$ unit = $\frac{1}{1000}$

Write this in a sequence, we have

$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10{,}000}, \frac{1}{100{,}000} \ldots$

This becomes unwieldy, so the Mathematician seeks to use a notation that is concise and neat. So he writes:

$1, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}, \frac{1}{10^5} \ldots \frac{1}{10^n}$

As n increases in value the terms get smaller and smaller and when n approaches infinity the tem approaches zero but will never equal to zero. Zero is called the limiting value of the sequence as n approaches infinity. Here zero is given another meaning.

The Greeks did not develop the above idea and this remained dormant until the seventeenth century when Newton and Leibniz independently developed the infinitesimal calculus using the above idea of limits.

(2) We return to the functions $f(x) = \frac{x}{x-1}$ or $y = \frac{x}{x-1}$. The mathematician wished to examine how this function behave for different values of x. He needs to get a pictorial view of this function. Is this possible?

Euclidian geometry has its limitation and many problems are not easy to investigate or solve by direct geometrical argument. The mathematician needs to invent another way to deal with geometrical problem. This Descartes did serve hundred years ago. From two number lines at right angle to one another he developed a grid system so the position of points and lines can be specified. Here is an example.

The two number lines are called the x-axis and y-axis respectively and with reference to values on these axes a point can be uniquely specified. Thus the point A (2,3) can be shown on this plane, called the Cartesian Plane after its inventor. The pair of numbers (2,3) are ordered, 2 is on the x-axis, 3 on the y-axis always in that order. Thus (3,2) is not the same as (2,3).

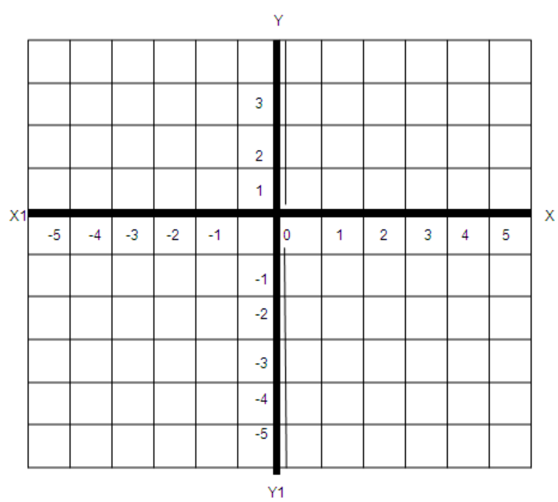

Fig. 4: Cartesian Plane

With this system, geometry can be converted to algebra and vice versa, proving itself to be a powerful tool for geometrical investigation. This is indeed an instance of superb creative thinking. The mathematician can use this idea to show a geometrical representation of y = $x/x-1$.

First he develops tables showing values of x and corresponding values of y.

Table 1: Values of x and y

A x approaches 1 from 0		B x approaches 1 from 2	
x	y	x	Y
0	0	2	2
0.25	-0.33	1.75	2.33
0.5	-1.0	1.50	3.0
0.6	-1.5	1.4	3.5
0.7	12.33	1.3	4.33
0.8	-4.0	1.2	6.0
0.9	-9.0	1.1	11.0
0.99	-99.0	1.01	101.0
0.999	-999.0	1.001	1001.0

Table 2: More values for x and y

A x approaches 1 from 0		B x approaches 1 from 2	
x	**y**	**x**	**Y**
0	0	3	1.5
-1	0.5	4	1.33
-2	0.66	5	1.25
-3	0.75	6	1.2
-4	0.8	7	1.16
-5	0.833	8	1.14
-6	0.85	9	1.125
-7	0.87	10	1.11
-8	0.88	11	1.1
-100	0.99	100	1.01

Geometrical representation of y = $1/x-1$ is obtained by plotting on the Cartesian plane the points shown on the above tables. The points are then joined. The mathematician now has a pictorial view of the function. It is easy now to observe how the function behaves, and important statements about the function can be made.

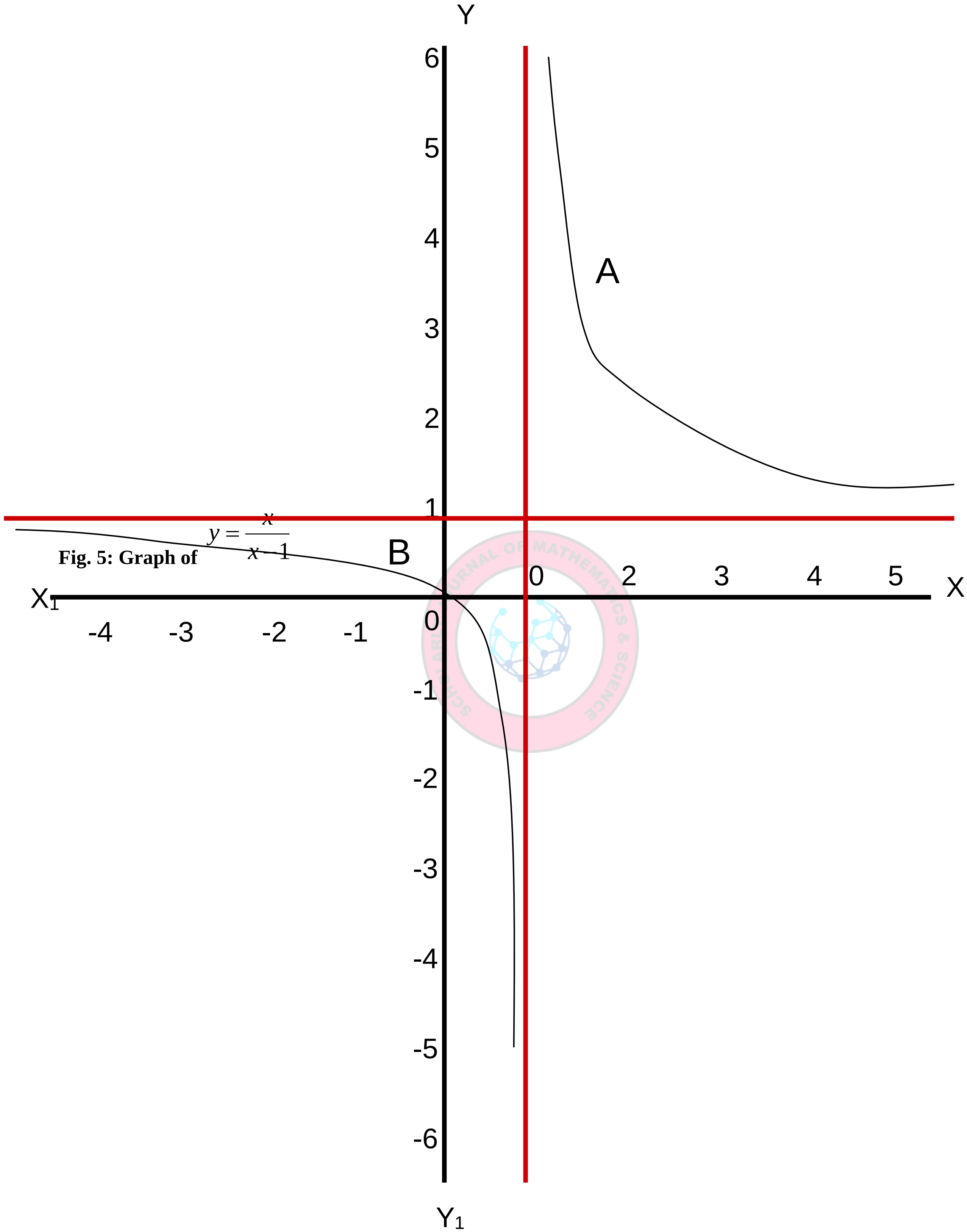

Fig. 5: Graph of $y = \frac{x}{x-1}$

An examination of the graph of the function shows that the function is not continuous: it has two points of discontinuity, and therefore two branches, here labeled A and B. We now refer to branch A. As x takes values from, say, 2 to 1 the function increases rapidly. As x approaches the vicinity of 1, (Here the denominator of y = $x/x-1$ approaches 0), the function approaches infinity, and if x = 1, the function is definitely infinite. We observe that as x gets close to 1, the function y increases rapidly and at the same time approaches the line x = 1 shown in red. The closer x approaches 1, the closer y approaches the line x = 1, but will not touch it. (This reminds us of the Achilles-tortoise paradox). If x =1, then at this point y is infinite and discontinues.

We now look at that branch of y = $x/x-1$ labeled B. As x approaches 1 from 0, we observe that y increases negatively. As x approaches the vicinity of 1, y approaches negative infinity and gets closer and closer to the line x = 1, but will not touch it. And if x = 1, y is negatively infinite and now discontinuous. The line x = 1 is said to be an asymptote to the curve represented by y = $x/x-1$.

The above discussion has important implications for that branch of mathematics called calculus.

The nature of the above function forces mathematicians to redefine zero. Zero is now a condition for rational functions, such as y, to become infinite at points, in this case the point x = 1 (on the x -axis). Mathematician say the functioning is singular at x =1.

The foregoing discussion shows that zero has two meanings in mathematics, each meaning being itself consisted but mutually incompatible frames of reference. In one case division by zero is not defined, and the other it defines a limiting value. It is from this ambiguous situation of zero, the mathematician has created new, powerful and beautiful mathematics.

We now examine again the branch A of the function, y= $x/x-1$. As x increases from 1 to infinity the curve gets closer and closer to the line defined by y=1, but will not touch it. At infinity the curve x = $x/x-1$ becomes discontinuous. Similarly the branch B of the curve defined by y = $x/x-1$ approaches the line y = 1 as x approaches negative infinity, but it will not touch y = 1. And at negative infinity the curve, branch B, becomes dis-continuous. The line y=1 is also asymptotic to the curve x = $x/x-1$. As x approaches positive infinity y approaches the value of 1, but will not exceed this value. Again as x approaches negative infinity, y approaches the value 1. Thus the mathematician says of the function y = $x/x-1$, that as x increases to infinity, both to positive and to negative, y approaches the value 1, and this value is referred to as the limiting value of y. Thus we see that this curve approaches infinity both positive and negative when x approaches the value 1, and when x goes to infinity, both to positive and to negative, y approaches the value 1.

This now takes us to the concept of limits, a vital idea in the development of the infinitesimal calculus. We look at the function y = $1/x$

x =2, y = $1/2$

x = 10, y = $1/10$

x = 100, y = $1/100$

$x = 10^{10}$, y = $1/10^{10}$

As x increases, y decreases. As x increases without limit, y decreases without limit, i.e. becomes so small that it approaches zero, and like the Achilles – tortoise paradox we ask if y will ever be equal to zero. But we continue . As x increases without limit, we say x approaches infinity. This is symbolized thus: $x \to \infty$, and at the same time y approaches 0, i.e. $y \to 0$.

In symbolic form we have when $x \to \infty$, $1/x \to 0$ or $y \to 0$($y = 1/x$).
Again when:

$x = ½$, $y = 1 / \frac{1}{2} = 2$

$x = 1/10$, $y = 1 / (1/10) = 10$

$x = 1/100$, $y = 1/(1/100) = 100$

$x = 1/10^{10}$, $y = 1/10^{10} = 10^{10}$ etc.

Again as x becomes smaller and smaller, y becomes large and larger. When x decreases without limit (it approaches 0) y increases without limit and it approaches infinity. Thus $x \to 0$, $y \to \infty$.

Let us take the function $y = 2x/x+1$. When x approaches infinity the function assumes the form ∞/∞. What meaning can be given to a rational function when it takes this form? One might be tempted to say 1 as ∞ cancels with ∞. But this is not so as ∞ is not a number with which we can operate. Multiplications or division of ∞ by any finite number leaves it still infinity. The mathematician does not easily accept defeat. He uses a legitimate strategy to overcome this dilemma.

Let's see:

$Y = 2x/x+1$

1. Divide each term of the numerator and denominator by x. We thus have $y = 2x/x \, / \, x/x + 1/x = 2/1 + 1/x$

This does not alter the value of the function.

Now $x \to \infty$, then $1/x \to 0$, so $y = 2/1+0 = 2$. Two is the value y approaches as x approaches infinity: y will not exceed this value and we say that 2 is the limit of y. The mathematician thus writes this as:

$$\text{Lt}_{x \to \infty} \; 2x/x+1 = 2$$

This is read as the limit of $2x/x+1$ as x approaches infinity is 2.

2. Let us examine the function $y = x^2 - 9/x - 3$.

The value of y is readily found for any finite value of x. But when x =3, y= $0/0$. Let us see.

a) Let x = 3.1, then $x^2 - 9/x - 3 = 9.61 - 0.61/0.1 = 6.1$

b) Let x = 3.01, then $x^2 - 9/x - 3 = 0.0601/0.01 = 6.01$

c) Let x = 3.0001, then $x^2 - 9/x - 3 = 0.006001/0.001 = 6.001$

We see that as x gets closer to 3, y gets closer to 6

a) Again let x = 2.9, then $x^2 - 9/x - 3 = -0.59/-0.1 = 5.9$

b) Let x = 2.99, then $x\,2 - 9/x - 3 = 0.0599/0.01 = 5.99$

Once more we see that as x gets closer to 3, y gets closer to 6.

The mathematician compares the results of the above and come to the conclusion that as x approaches 3, y approaches 6, and ultimately when the value of x differs from 3 by an infinitely small number, the value of y also differs from 6 by an infinitely small number. He symbolizes this conclusion thus:-

$x \to 3,\ x\,2 - 9/x - 2 \to 6$

The functions $y = x^2 - 9/x - 2$ has a limiting value as x approaches 3 or using the notation for limits.

$$\operatorname{Lt}_{x \to 3} x^2 - 9/x - 3 = 6$$

When x =3, y assumes the indeterminate form $0/0$. This does not defeat the mathematician. He works like a scientist, experimenting with the function $y = x^2 - 9/x - 3$ by assigning values to x and observing the results on the basis of which he makes a conclusion. The mathematician is not satisfied with this effort; he needs to generalize to arrive at general truths. So he proceeds thus:

Let $y = x^2 - a2/x - a$. If x =a, y takes the indeterminate form $0/0$. Using the same thread of argument as above, he argues this way:

Let x = a + h… (1).
Then $x - a = h$, h being the difference between x and *a*. For different assigned values of x, h will vary. Refer to the experiment above. Using (1) and substituting for x in $x^2 - a2/x - a$. We have :

$$x^2 - a^2/x - a = \frac{(a+h)^2 - a^2}{(a+h) - a} = 2ah + h^2/h = 2ah + h$$

Using a technique the mathematician used earlier, we divide the terms of the numerator and denominator by *h* which is not zero. We have

$$X^2 - a^2/x - a = \frac{\frac{2ah}{h} + \frac{h^2}{h}}{\frac{h}{h}} = 2a + h \ldots(2)$$

From (1) where x -a=h, as x gets nearer to a, h gets smaller as h is the difference between x and a. Thus when x approaches infinitely near to a, h approaches zero, so that from (2) 2a + h approaches 2a. The mathematician says this symbolically, when $x \rightarrow$ a, h $\rightarrow$ o, the function x^2-a²/x -a $\rightarrow$ 2a, i.e. 2a is the limiting value of the functions when approaches a. Using the notation for limits, we write the above statement like this

$$\underset{x \rightarrow a}{\text{Lt}}\ x^2 - a^2/x - a = 2a$$

The mathematician uses the expression $0/0$ to represent the ratio of two infinitely small magnitudes. The value of this ratio approaches a finite limit as numerator and denominator approach zero. This new role of zero prepares a fertile spawning ground for the development of a most powerful branch of mathematics – the calculus.

Let us see how this is done. Examine the graph of the function y = x^2 (Fig. 5). This is a curve. The line AB intersects this curve at P and Q respectively. This line is called a secant to the curve and PQ is called a chord of the curve. Let the x – coordinate of P get an increase δx which takes it to R whose coordinates are ($x + \delta x$, y). But if x get an increase, y will also get an increase, say δy and this takes it to Q whose coordinates are $x + \delta x$ and y + δy .

The gradient of the line AB is defined as the ratio $\delta y / \delta x$. As δx decreased so does . When this happens Q moves along the curve to a new position and Q_1 and the line AD rotates about P and PQ gets shorter. As δx gets shorter and shorter, so does δy ; and Q moves along the curve to approach the point P, while at the same PQ gets shorter and shorter as AB rotates about P. As $\delta x \rightarrow$ 0, Q approaches P, but will not go beyond P, and at the same time the chord PQ $\rightarrow$ 0.

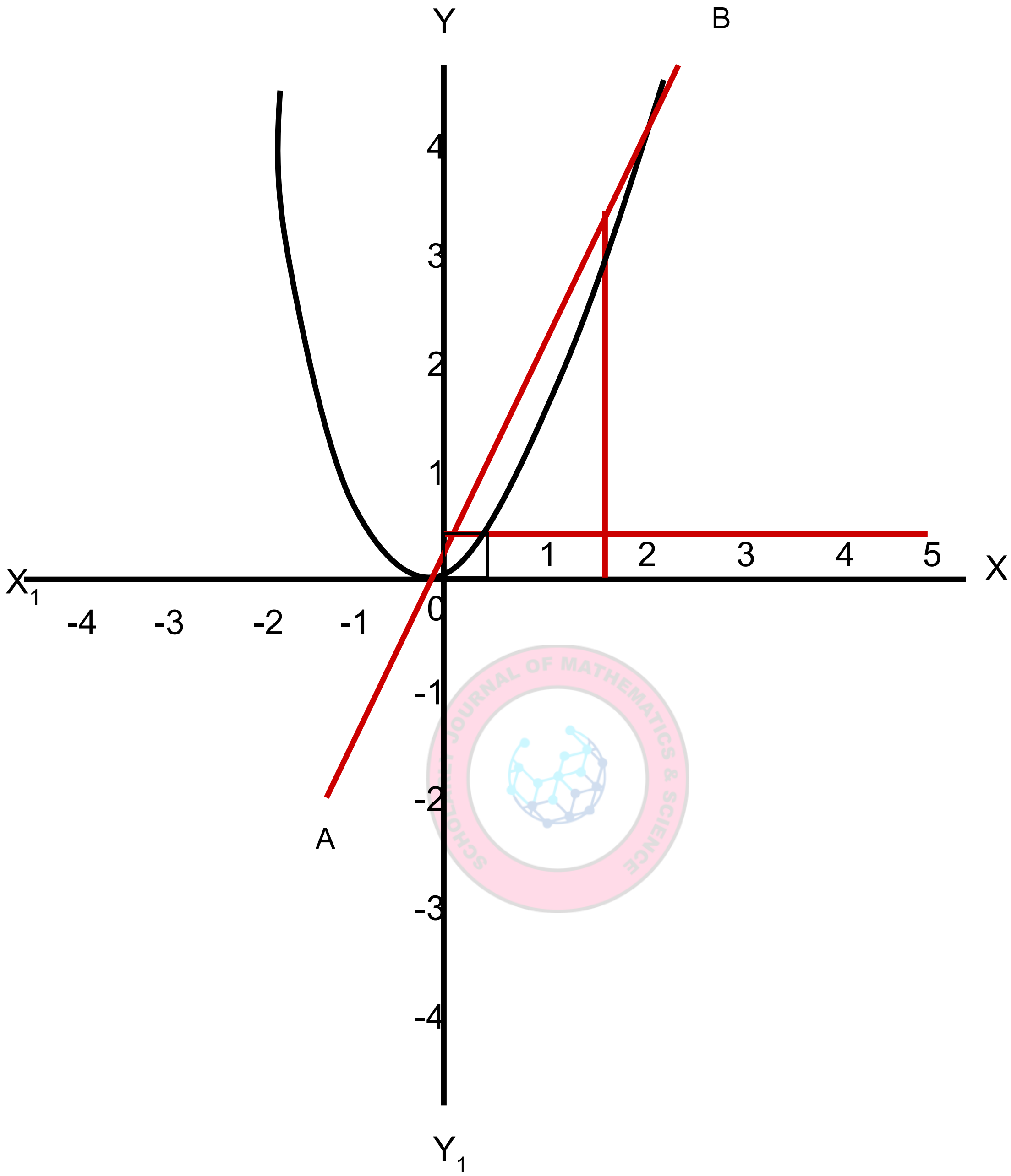

Fig. 6: Graph of $y = x^2$

Thus the gradient of PQ becomes the gradient of the curve at P and the line at AB now touches the curve at only one point P and so by definition AB is a target to the curve at P and has the same gradient as that of the curve at P.

The gradient of the chord PQ is $\delta y / \delta x$ and the limit of $\delta y / \delta x$. As $\delta x \rightarrow 0$ is the gradient of the curve at P and consequently the gradient of the tangent of the curve at P. The above discussion can be translated algebraically as follows:

y= x^2 (1)

If x gets an increase, say δx, then y will get an increase δy.

$\therefore$ y + $\delta y = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$. (2)

We need to examine only the increases in y and x. So we find the difference between (2) and (1) and we now have

$\delta y = 2x\delta x + (\delta x)^2$ (3)

Thus the difference between the augmented value of the function and the function itself is $\delta y = 2x\delta x + (\delta x)^2$.

From the above we see that the limit of $\delta y / \delta x$ as $\delta x \to 0$ is the gradient of the curve a given point. So we divide (3) by δx to obtain the ratio $\delta y / \delta x$. We have

$\delta y / \delta x = 2x + \delta x$ (4)

Thus $\lim_{\delta x \to 0} \delta y / \delta x = 2x + 0 = 2x$ (5)

Now , but it is not zero, yet in equation (4) zero is submitted for δx. This is contradictory!

We refer to LT $\delta y / \delta x$ as dy / dx. So if y = x^2.
$\delta x \to 0$

$dy / dx = 2x$ (6)

The symbol dy / dx represents a limit and $\delta x \to 0$ is understood. The infinitely small increase in x and y are represented by dx and dy respectively and these are called differentials. Thus dy / dx represents the ratio of the differential of y to the differential of x and this is 2x.

So from (6) we see the differential of y is 2x times the differential of x, i.e.

dy = 2x dx

In this form 2x is the coefficient of dx which is the differential of x, and hence 2x is called a differential coefficient.

We see earlier that dy / dx measures the gradient of the chord PQ and that the ratio dy / dx measures the gradient of the curve at the point x and incidentally this is also the gradient of the tangent at that point.

Assign different values for x in $dy / dx = 2x$.

when x = 1, $dy / dx = 2$

when x = 2, $dy / dx = 4$

when x = 3, dy/dx = 6 etc.

From the above we see that the ratio dy/dx called the differential coefficient or the first derivative of the function of x is twice any assigned value of x, i.e. it increases or decreases twice as fast as x, hence we consider dy/dx as a rate measure. This has important implications in science, astronomy and economics. The distance a body moves is a function of its velocity and time. The differential coefficient of this function with respect to time defines the velocity of the body for any given period of time. Thus the mathematician uses the ideas of the infinitely small and redefines zero as a limiting value, and the infinitely big where curves approach a line(s) but will not touch it or cross it as also a limit – the limit as infinity to create a most powerful branch of mathematics – the calculus. Calculus germinated from a paradox, nurtured by ambiguity and contradictions, and once considered by the French mathematician, Michael Rolle, as "a collection of ingenious fallacies:, has blossomed into a full blown discipline. Calculus is the chief interpreter and arbiter of science and astronomy; it provides the right answers to problems or questions in these disciplines and it works well, but why.

3.0 DEDUCTION – A POWERFUL TOOL OF THE MATHEMATICIAN

Deduction is a logical thinking process that is involved in drawing out a truth from a pre-established truth. This new truth forms the basis from which other truth(s) can be deduced. In order to have a truth or a premise, it is important to note that the Greek mathematicians started with general premises they called axioms and more specific premises called postulates. An axiom is a premise whose validity is self-evident and therefore cannot be subjected to the rigors of proof. On the other hand postulates are less assured premises. While axioms and postulates form the basis for deductive reasoning, they do not span much of the world of truths. New truths beside axioms and postulates must be discovered to promote the growth and development of mathematics. Another kind of reasoning is required to establish valid premises for deductive reasoning. This is inductive reasoning. The first part of this paper dealt with an example of inductive thinking. We call it here. It is a set of natural numbers: {2, 4, 6, 8, …}. The mathematician studies these numbers with a view to discover any common characteristic of them. He finds that a member in the set it a multiple of 2, and they form a sequence in a certain way that makes it easy to predict other numbers of this kind in a given sequence like the above.

The mathematician calls these numbers even numbers and he generalizes that all even numbers are in the form 2n, where n is a natural number. This is a premise derived from induction and whose truth is unquestionable. This premise can now form the basis of deductive reasoning thus:

All even numbers are in the form of 2n
62 is in the form of 2n i.e. (2 x 31), where 31 replaces n.
∴ 62 is an even number

Here we see that there was no need to iterate all the even numbers in the sequence {2, 4, 6, 8,…} to find out that 62 is a member of this set. Deduction does the job very economically.

Again
All even numbers are in the form 2n
93 is not in the form 2n
∴ 93 is not an even number

We now turn to Geometry and examine how the mathematician uses deductive reasoning to establish new truths from pre-established truths.

Theorem: The medians of a triangle are concurrent, i.e. the medians intersect at one point. A median is a straight line drawn from the vertex of a triangle to the point of the side opposite the vertex.

To show the truth of the above theorem, the mathematician constructs a triangle ABC, and then constructs two medians. CZ and BY . (Fig. 5)

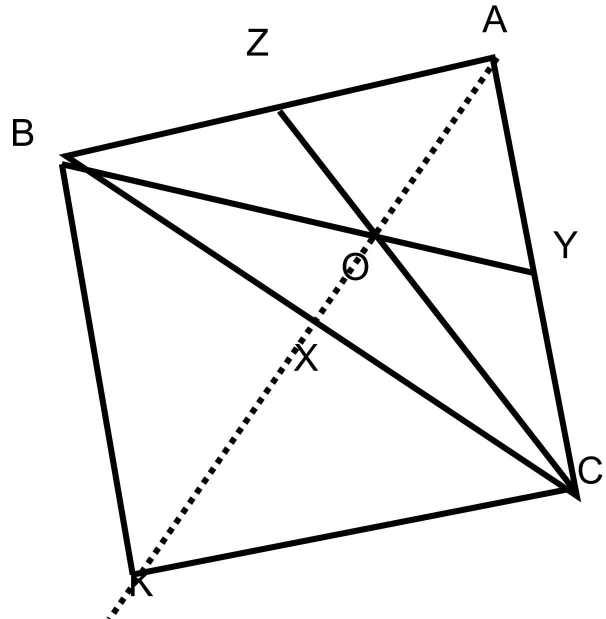

Fig. 6: Medians of Triangle ABC

The mathematician knows that these two lines can intersect at once and only one point, say). Here the axiom that two straight line can intersect at only one point assures that mathematician of this situation. He joins AO and produces it to X, a point on BC. SO he constructs AX in such a way that it passes through the point of intersection of the medians CZ and BY. Now he needs to show that AX is a median. To do this he has to prove that X is the mid-point, viz. the diagonals of a parallelogram bisect each other. But there is no parallelogram in △ABC. So he constructs a quadrilateral with BC and OK as diagonals. He proceeds thus: Through C draw a lien parallel to BY, and produce AX to meet it at a point K. Now join BK. The mathematician has to prove BKCO is a parallelogram. Let us define 'proof in maths'.

"A poof in maths is an impeccable argument using only the methods of pure logical reasoning which enables one to infer the validity of a given mathematical assertion from pre-established validity of other mathematical assertions" (Penrose, 2006). The mathematician furnishes a proof with each step of reasoning being as economical as possible, precise and logically sound. We need to 'look into the mind' of the mathematician to use how he thinks, the strategies he uses and the ways he uses established truths to prove the validity of given mathematical assertions. Back to the problem of medians.

The mathematician has to prove that the quadrilateral BKCO with diagonals OK and BC is a parallelogram. He has constructed CK to be parallel to BO. Now he has to show that OC is parallel to BK.

He studies the above diagram carefully and focuses on the lines BY and CK which are parallel by construction. He observes that CK is the base of the △AKC and that OY is parallel to it. He gives him the clue that he can use the theorem which asserts that if the intercepts made by parallel lines on any transversal (a transversal is a straight line that cuts two or more lines) are equal. The converse of this theorem is also true, viz., if the intercepts made by two or more lines by two or more transversals are corresponding equal, then the two or more lines are parallel. He proceeds thus:

In the △AKC,
Y is the mid-point of AC by definition of median. Thus AY and YC are equal. Since OY and CK are parallel, AC is considered a transversal and so is AK. As already given AY = YC on the transversal AC, then AO = OK on the other transversal AK.
Thus O is the mid-point of AK.

In the ΔABK,

Z is the mid-point of AK, already proven.
Thus the intercepts made by the lines CZ and BK on the transversal AB are equal, i.e. AZ = BZ.
Again the intercepts made by the lines CZ and BK on the transversal AK are equal, i.e. AO = KO, already proven. The CZ is parallel to BK, (converse of intercept theorem) i.e. CO is parallel to BK.

Hence BKCO is a parallelogram

3.1 Conclusion

In the parallelogram BKCO, OK and BC are its diagonals, and it is known from a previous theorem that diagonals of a parallelogram bisect each other, and in this case OK and BC bisect each other in X. Therefore X is the mid-point of BC, and by definition of median AX is a median, passing through O, the same point the medians BY and CZ pass through. This now concludes in the theorem that the medians in any triangle are concurrent. (The point O is also called the centroid of the triangle and this concept is used in Mechanics to determine the centre of mass of a triangular lamina).

The mathematician studies the triangle ABC and the construction he makes on it to see if new truths or theorems called corollaries can be drawn. He observes that:

KX = OX because X is the mid-point of OK
But OK = AO already proven

If KK = 1 unit (of measure) so is OX because KK = OX.
Now KK + OX = 1 unit + 1 unit = 2 units.
But KK + OX = OK
Thus OK = 2 units
But OK = AO
Then AO = 2 units, and from above
OX = 1 unit, so that AX is 3 units
OX then is $\frac{1}{3}$ of AX
It can similarly be shown that
OY is $\frac{1}{3}$ of the median BY and
OZ is $\frac{1}{3}$ of the median CZ

Thus the three medians of the triangle cut one another at the point of O. This point O is called the point of trisection of the medians. Since ABC is any triangle the mathematician generalizes by stating that the medians of a triangle cut one another at a point of trisection, i.e. the three medians are concurrent at a point of trisection, i.e. the three medians are concurrent at a point of trisections. This means that if the medians AX, BY and CZ are each divided into three equal parts by points on them, then o is the common point on them dividing each in the ratio 2:1.

This theorem can be proved very elegantly by the use of vectors.

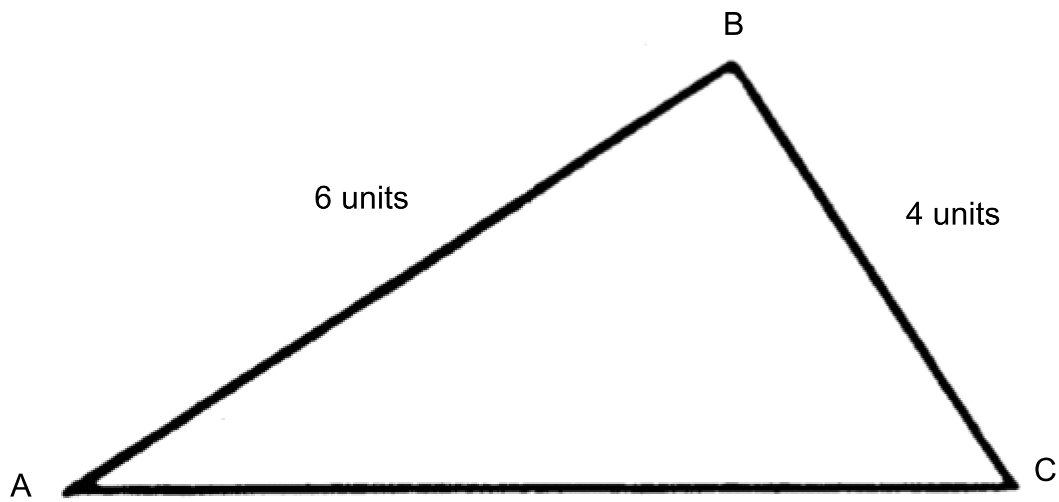

Fig. 7: A triangle of vectors

If a man walks from A to B a distance of 6 units, and then from B to C a distance of 4 units Fig. 7, we do not hesitate to conclude that he has walked a distance of 6 units + 4 units = 10 units, the algebraic sum of 6 and 4. In vector addition this is different, i.e. 6 units + 4 units is not the same as 10 units in the above situation. If, however, AB and BC are segments of the straight line AC, then the algebraic sum is considered valid.

A vector is an entity in Euclidian space (a plane flat surface) that has both magnitude and direction. A boat travels at a speed of 10 knots per hour in an easterly direction. The mathematician represents this on paper by a line segment, 10 units long (the magnitude), drawn eastwards (the direction) with reference to a directional frame.

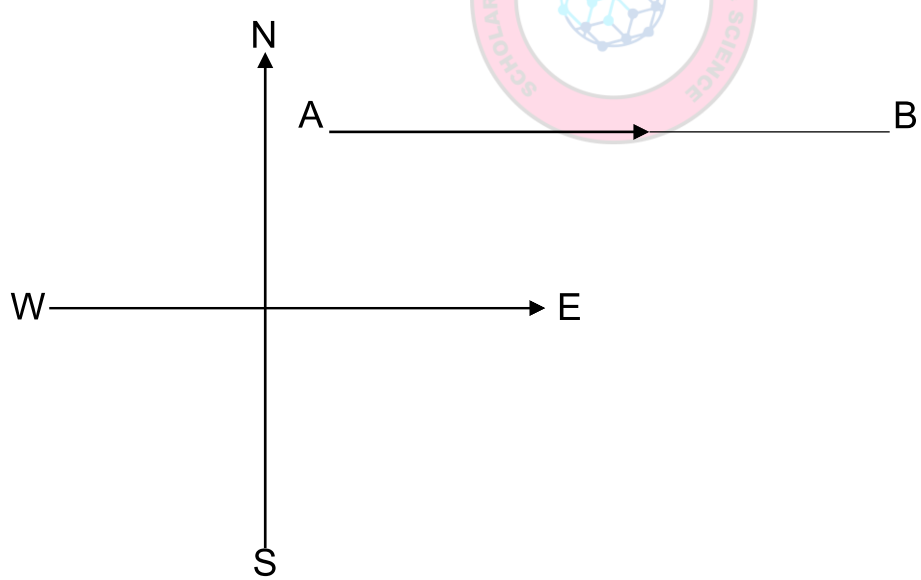

Fig. 8: Directional frame

AB is the vector representing the velocity, and also the distance travelled from A to B in one hour. If we write the vector as BA, this indicates that the boat is still travelling at 10 knots/hour, but in the opposite direction viz. in westwards. Thus BA = -AB. The distance one travels in a given direction and the velocity of travel in a given direction are represented by vectors.

To show that in any triangle ABC, the medians are trisected by the point of concurrence.

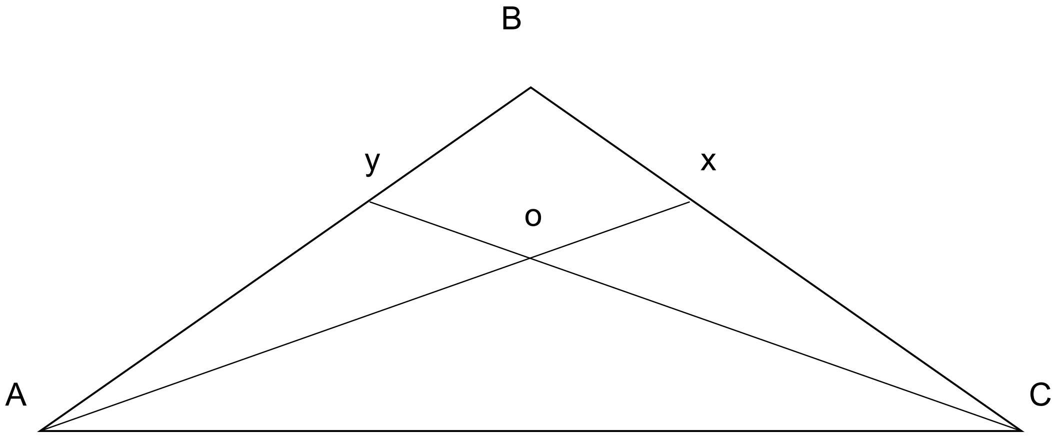

Fig. 9: Medians of a triangle

The mathematician now recalls a fact of vector addition, which he will use in the proof.
Thus AB + BC = AC, i.e. if one moves from A to B and then from B to C, it is just like one moves from A to C (Fig. 9). The mathematician writes this in a more convenient form; he represents AB by b and AC by c. Then the above equation can now be written as a+b=c. CO is a fraction of the median CY and so is AO a fraction of the median AX. He now sets up an equation that shows the relation of the vectors CO and CY. Thus:

CO =mCY, where m represents the fraction. He needs now to represent CY in terms of a, b, and c.
CY = CA + AY (Vector addition)
But CA = -CA (change of directions)
So CY = -AC + AY
∴ CY = -c + ½a. (AY = ½ AB, as Y is the mid-point of AB).
Now c = a+b
∴ CY = -(a + b) + ½a = - ½a – b
∴ CO = m(-½a –b)

Again
AO = nAX, where n represents the fractions. The mathematician represents AX in terms of a, b, and c.
Thus AX = c- $\frac{1}{2}b(CX : \frac{1}{2}b$, as X is the mid-point of BC).
But $C = a + b$, so that
AX = a+b - $\frac{1}{2}$ b = a = $\frac{1}{2}$ b
$\therefore nAX = n(a + \frac{1}{2}$ b)
In the △OAC, AO = AC + CO.
Thus from (2) and (1)
n(a + ½b) = c + m (-½a –b)

Writing c in terms of a and b to reduce the number of variables, we have
n(a + ½ b) = a + b + m (-½ a –b)
The mathematician now compares a and b by putting one of them on the right hand side of the equation and the other on the left. Thus, we have a
(n-1 + ½m) = b(1 - ½n-m).

Since a and b are not equal vectors, the equation can be true if
(n – 1 + ½m) = 0 and

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$(1 - ½n – m) = 0$.

These equations are solved simultaneously given the solution

$m = \frac{2}{3}$ and $n = \frac{2}{3}$

The mathematician interprets these as CO = $\frac{2}{3}$ CY, i.e. CO is $\frac{2}{3}$ of the medians AX and CY. Similarly AO = $\frac{2}{3}$ AX, i.e. AO is $\frac{2}{3}$ of the median AX. O then is the point of trisection of the medians AX and CY. By similar argument the mathematician can show that O is the point of trisection of the third median. This establishes the theorem that the medians of any triangle trisect one another as the point of concurrence.

3.2 Proof of Contradiction

The proof of the converse of a theorem is often done by an assumption that is contrary to the theorem. For example, if two angles of a triangle are equal then the sides opposite to the equal angles are also equal. This is the converse of the theorem which asserts that the angles at the base of an isosceles triangle are equal. By definition an isosceles triangle has two equal sides. This is sufficient to prove that the base angles are equal. The mathematician proceeds as follows:

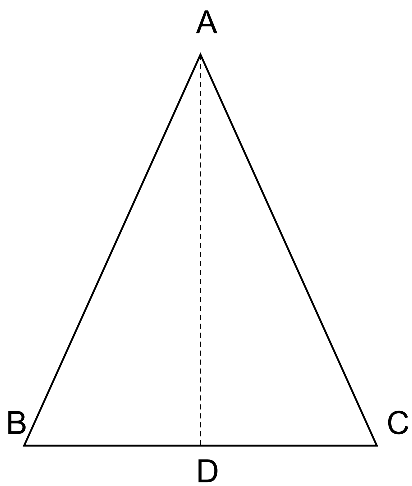

Fig.10: The isosceles triangle

In the triangle ABC (Fig. 10) construct a line AD which bisects <BAC and meets the side BC and D. Thus constructing the line AD, the mathematician prepares the necessary conditions for congruency of triangles. He then argues that triangle ABD and triangle ACD are congruent, i.e. the triangles are equal in all respects, so that <ABC = <ACB. The converse of this theorem is already above, but to recall, it states that in a triangle if two angles are equal, then the sides opposite the equal angles are also equal. Here the mathematician cannot make a construction to make the necessary condition to invoke relevant theorems, nor is there a theorem from which he could deduce the truth of the above theorem directly. But he does not give up easily. He proceeds like this:

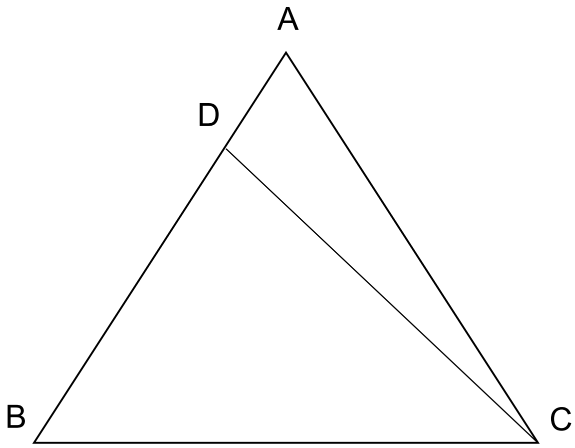

Fig. 11

In the triangle ABC (Fig. 11) <ABC = <ACB.
The mathematician assumes that AB is not equal to AC, and that AB is greater than AC.
He selects a point D on AB so that DB = AC. Join CD.
Then, he argues, in the △'s DBC, ACB,

DB = AC (assumed)

BC is common to both
And the included <DBC = the included <ACB
But these are the conditions of congruency of triangles. It must be noted that the mathematician is using a condition of congruency by assuming AB is not equal to AC, and that DC is equal to AC. This brings too conclusion that

The △ DBC = the △ ACB in area, i.e.,

the part is equal to the whole. This is absurd and not true.
Then AB is not unequal to AC, i.e. AB = AC.

This is proof by contradiction by showing the assumption is absurd. This is a case or reducing the assumption to absurdity – reduction ad absurdum.

Here is another proof by contradictions – this time in Algebra.

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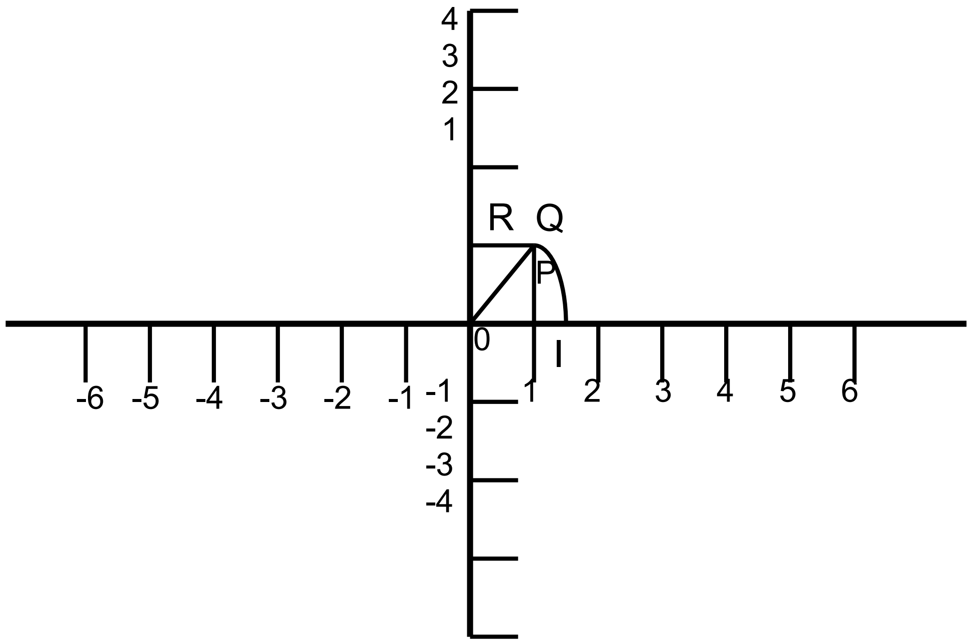

Fig. 12 Diagram showing $\sqrt{2}$

The square in the above diagram Fig. 12, has a side of 1 unit. By the theorem of Pythagoras $OQ^2 = OP^2 + PQ^2$ i.e.

$OQ^2 = 1^2 + 1^2 = 2$

$\therefore$ OQ = √2

It has been found that √2 is not an integer, nor is it a rational number, i.e. a number in the form P/q, where both p and q are integers and q is not zero. This number cannot therefore be represented by an integer or by a rational number, but a point on the number line marks its position. In the diagram the point I marks the position of √2. This number can be represented geometrically and is therefore a real number. It lies between 1 and 2. It has been agreed by mathematicians to call √2 Irrational, and is indeed an algebraic number as it can be the root of a polynomial equation. Thus if

$X^2 – 2 = 0$

$X^2 = 2$

Whence $x = \pm\sqrt{2}$

√2 is a root of the equation $x^2 – 2 = 0$

<u>To show that √2is irrational. A proof by contradiction. Preliminary considerations.</u> The mathematician invokes the definition for even numbers. See first part of this paper.

An even number is a multiple of 2. Any even number can be written 2n, where n is any integer. If p and q are even, the p/q, is a fraction or a rational number not in its lowest term, as 2 is a common factor of both p and q.

The mathematician assumes √2 is rational. So he states that √2 = p/q, where p, q are integers with no common factor, and q is not zero, i.e. p and q are co-primes.

The mathematician now performs a mathematically legitimate operation; he squares both sides of the equation to get $2 = p^2/q^2$ or $2q^2 = p^2$, and puts up the following argument. But $2q^2$ is a multiple of 2 and therefore p^2 is even and so is

p. Therefore p is in the form of 2n, so $p^2 – 4n^2$. Thus p^2 contains the factor 4 and the above equation can be written thus $2q^2 = p^2 = 4n^2$, i.e.

$2q^2 = 4n^2$

$\therefore q^2 = 2n^2$

$\therefore$ q is even

But p and q are even, which means they have a common factor. But p and q were assumed to have no common factor, and this assumption has been contradicted by the above reasoning. Therefore √2 cannot be represented by $^{p}/_{q}$ which is a fraction or rational number. Then √2 is not a rational number, it is an irrational number.

Proof by Exhaustion

This form of proof is applicable to cases in which one of a number of supposition must necessarily be true. It shows that all the suppositions with the exception of one are false. The truth of this one is inferred.

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less. Here is a triangle ABC Fig. 13. The angle ABC is greater than the angle ACB.

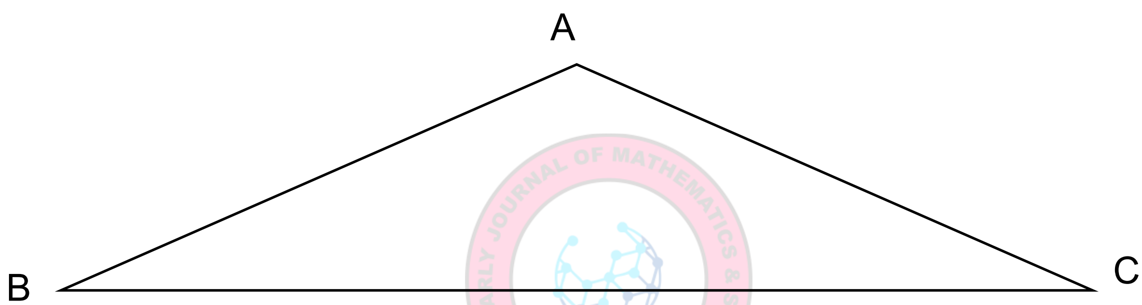

Fig.: 13

The mathematician has to prove that AC is greater than AB. He argues like this:

If AC is not greater than AB, then it must be either equal to, or less than AB. If AC were equal to AB, then <ABC would be equal to the ,ACB. This follows from a previous theorem on isosceles triangle. But by hypothesis (stated in the above theorem) this is not so.

Again if AC were less than AB. Then <ABC would be less than <ACB. This follows from a pre-established theorem. But by hypothesis this is not so. That is, AC is neither equal to, nor less than AB. Therefore AC is greater than AB.

Proof by Induction

The set of natural number N is given as:

N = {1, 2, 3, 4, …n}. Check the first paragraph of this paper}.

What is the number of the first n natural numbers. The mathematician proceeds like this. He writes S_n to represent the sum. Thus

$S_n = 1 + 2 + 3 + 4 + \ldots n$ (1)

There is no way the mathematician could generalize from this statement. From experience he knows that

1 + 2 + 3 + 4 + 5 = 5 + 4 + 3 + 2 +1. That is the order of addition does not change the value of the sum. Now if S_5 donates the sum of the first five terms, then a statement about this can be written

Thus $S_5 = 1 + 2 + 3 + 4 + 5$ (a)

Or $S_5 = 5 + 4 + 3 = 2 + 1$ (b)

Looking carefully at (a) and (b) the mathematician gets

$S_5 = 1 + 2 + 3 + 4 + 5 \ \ldots$ (a)

+

$S_5 = 5 + 4 + 3 + 2 + 1$ (b)

$2S_5 = 6 + 6 + 6 + 6 + 6$. This is twice the sum of either (a) or (b).

So $S_5 = \dfrac{6+6+6+6=6}{2} = \dfrac{5x6}{2} = 15$

This gives the mathematician a clue how to tackle (1) from above

$S_n = 1 + 2 + 3 + 4 + \ldots \quad n \quad \ldots$ (1)

And $S_n = n + (n-1) + (n-1) + n-2) + (n-3) + \ \ldots \ 1$ (2)

By adding (1) and (2), he gets

$2S_n = (1 + n) + (n + 1) + (n + 1) + (n + 1) \ldots (n + 1)$

Thus $2S_n = (1 + n)$ added to itself n times

$\therefore 2S_n = (1 + n)n$

$\therefore 2S_n = \dfrac{(1+n)n}{2} \ or \ \dfrac{n(n+1)}{2}$

While this formula has been deduced from first principles, it is necessary to prove its validity. There is no pre-established theorem which the mathematician can use to deduce the truth of the formula. He, however, uses another approach, viz. mathematical induction. He assumes that

$Sn = \dfrac{n(n+1)}{2}$ is true

He adds one more term to the n terms, and so now there are n + 1 terms, the (n + 1) the term being n + 1. So adding (n + 1) to the formula,

we have $S_{n+1} = \dfrac{n(n+1)}{2} + \dfrac{n+1}{1}$

$= (n+1) \dfrac{n}{2} + 1 = \dfrac{(n+1)(n+2)}{2}$

This is the same formula as above where n is now replaced by (n+1).

If the formula is true of n terms, then it is true for (n+1) terms. Other values for n is used and the formula tested.

When n = 2, $S_2 = \dfrac{2.3}{2}$ = 3, i.e. the sum of 1 + 2 + 3 (two terms in the series). When n = 3, $S_3 = \dfrac{3.4}{2}$ = 6, i.e. the sum of 1 + 2 + 3 = 6 (three terms in the series). When n = 4,$S_4 = \dfrac{4.5}{2}$ = 10, i.e. the sum of 1 + 2 + 3 + 4 = 10 (four terms in the series) and so on.

The formula holds good for 2 terms, 3 terms, 4 terms and so on. It holds for (n +1) terms. Therefore 1 + 2 + 3 + ….+n $= \dfrac{n(n+1)}{2}$, that is $Sn = \dfrac{n(n+1)}{2}$ is true as assumed.

4.0 MATHEMATICAL LOGIC

Symbolic logic is the study of various types of deduction in general, and since this type of logic employs mathematical symbols, we refer to it as mathematical logic. Logic is an idea. It is organizational principle par excellence in mathematics and science (Byers, 2007). It stabilizes mathematical ideas, organizes mathematics into structures that promotes development of theoretical mathematics and provides the tools for communicating mathematical ideas precisely and succinctly. The mathematician deals with statements. Here are a few statements.

(1) The sum of the internal angles of any quadrilateral in a plane is 360°.
(2) If the angles of a triangle are equal to the angles of another triangle, each to each, then the triangles are also equal in area.
(3) There exist an integer x such that x is even and prime.
(4) There exist a number such that a°=O.

Statement (1) is true. This is a well-known result in plane geometry. Statement (2) is false as it is not a condition for congruency. If one side of a triangle is equal to a corresponding side of another triangle and their angles are equal, then the triangle is congruent and their areas are equal. So statement (2) can only be true if another condition exists, i.e. at least one side of one triangle must be equal to a corresponding side of another triangle. Equality of the angles, at least two, and a corresponding side of the triangles are the sufficient and necessary conditions for congruency. (3) This is true. The integer 2 is both a prime and even number. (4) This statement is false. The laws if indices establish that a°=1. From the above discussion we see that statements are either true or false. If a mathematical statement make sense, it is either true or false; it cannot simultaneously be both. This is a fundamental law of thought.

Let us look at this equation, x + 6 = 5. Is it a statement? A statement is either true or false. We cannot determine whether the equation is true or false, so as the equation stands, it is not a statement. Mathematicians call x + 6=5 an open sentence. If x replaces by a specific number, say 3, then the equation becomes 3 + 6 = 5. But this is false. Once the falsity or truth of the equation is established, it is considered a statement. Replacing x by a set of numbers will create false statements, with the exception of one number which will make the equation true. What is this number? Is it -1? Then in the equation replacing x by -1, we have -1 + 6 = 5. This is true, and we see that x in this case can be replaced by only one number, viz. -1. This makes the open sentence x + 6 = 5 a statement. Now one equation like 2 x = x = 3 x is true for all values of x but is still an open sentence, since no specific value of x is indicated. But if x =1, then from the equation 2 x + x = 3 x, we have 2 x 1 = 1 = 3 x 1 = 2 + 1 = 3, which is true, so 2 + 1 = 3 is considered a statement. If x = 2, 2 x + x = 3 x becomes 2 x 2 + 2 = 3 x 2, i.e. 4 = 2 = 6. Again a true statement, and so on for all number replacement for x.

From given statements new statements can be formed by the use of 'and', 'or', 'not'. Statements so formed care called compound statements.
Here are two statements.

(1) The integer a satisfies a is greater than 2.
(2) The integer a satisfies a is less than 9.

We can form a compound statement by joining (1) + (2) by 'and.

(3) The integer a satisfies a is greater than 2 and is less than 9. Many people are comfortable with this statement and can reason with it quite comfortably. However, there is a limit to this. Where proofs are required and where many variables are to be dealt with, statements like the one above need to be written in a concise form, particularly where mathematical operations are to be performed on the variables. Here the mathematician steps in.

The mathematician symbolizes the statement. For 'and' he uses $\wedge$ and for 'or' he uses $\vee$. He replaces statement (1) by p and (2) by q, 'greater than' and 'less than' by > and < respectively. Statements (3) can now be written like this: p $\wedge$ q. This of course is interpreted as a satisfies 2 < a < 9.

We now look at the way the mathematician uses $\vee$. While $\vee$ stands for 'or', in mathematics, it is almost always used in the inclusive sense, i.e. p $\vee$ q means p or q or both.

If p is a statement, 'such that 7>6', the mathematician writes this as p: 7>6, where : replaces 'such that', and q: 9<10.

Here p ∨ q means that 7>6 or 9<10. Both statements are true, therefore p ∨ q is true.

If p: 3>4; q : 1>0, the p ∨ q means that 3>4 or 1>0. Here p is false and q is true. Therefore p ∨ q is true since q is true, i.e. if either p or q is false but not both, then p ∨ q is true. If p and q are both false, then p ∨ q is false, then p ∨ q is false.

If p is a statement, then by prefixing the word ‘not’ to form ‘not p’, we have so formed the negation on p, denoted by ⊔ p is false, and if ⊔ p is true, then p is false. The statements p and ⊔ p cannot be true. Thus

(1) p: the angle in a straight line (straight angle) is 180° :- true ⊔ p : the angle in a straight line is less than 180° :- false.
(2) p : For each pair of consecutive integers x, y, x + y is even: false ~p: There is a pair of consecutive integers x, y, such that x + y is odd: true.

A statement that is true has a truth value T, and if false, a truth value F. The truth as falsity of a compound statement can be determined by the truth or falsity of its components which are p and q. The mathematician neatly summarizes the above idea in a truth table for ⊔ , ∧ and ∨ .

Table 3

P	⊔ P
T	F
F	T

Table 4

P	q	p ∧ q	p ∨ q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Truth tables can be drawn up for more complicated compound statements such as ⊔ (p ∧ q) or (⊔ p) ∨ (⊔ q) or (p ∧ q) ∨ (⊔ p) etc.

Table 5

p	q	r	q ∨ r	p ∨ (q ∨ r)	p ∨ q	p ∨ (q ∨ r)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T
F	F	F	F	F	F	F

Table 3-5 : Truth tables for Compound Statements.

From the above table, we see that the columns for p ∨ (q ∨ r) are identical to the truth values for $(p \vee q) \wedge (p \vee r).$ We conclude that $p \wedge (q \vee r) = (p \wedge r).$ This is analogous to the distributive property of x over + in number algebra i.e. $a(b+c) = ab + ac$. Thus ∧ is distributive over ∨. Similarly we see that $p \wedge q = q \wedge p$, i.e. ∧ is commutative.

If p is an even number and q is an odd number, then p $\wedge$ q stands for "an even number and an odd number". (1)

q $\wedge$ p stands for "an odd number and an even number". (2)

Statements (1) and (2) are equivalent and p $\wedge$ q=q $\wedge$ p is established. It can be shown that $p \vee q = q \vee p$. Thud $\vee$ and $\wedge$ are commutative property of number algebra for + and x we have $a+b=b$ or $b=b+a$ or $ab=ba$.

The mathematician observes the similarities of the properties of $\wedge$ and $\vee$ in logic to those of $+$ or x in number algebra and since $\wedge$ and $\vee$ are more general than $+$ or x he uses this logic to organize and stabilize mathematics and to control ambiguity.

In number algebra we deal with the variable 'x', but the variable 'not x' is not a variable in algebra. This somewhat limits the scope of number algebra. The mathematician recognizes the power of logic and takes advantage of this. And in so doing discovers new truths. Here is another truth table.

Table 7: Negation of propositions

p	q	~q	~q	p^q	~(p^q)	~p∨~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The above table shows that the truth for ~(p^q) and ~p∨~q are the same and so ~(p^q) = ~p∨~q. From these simple ideas the mathematician builds a complex body of mathematical knowledge.

If the statement q is so related to p that the truth of q follows from the truth of p, we say that p implies q i.e. if p, then q, and this is denoted by p $\Rightarrow$ q. If q $\Rightarrow$ p, this is the converse of p $\Rightarrow$ q. If p $\Rightarrow$ q and q $\Rightarrow$ p, this is shown as p $\Leftrightarrow$ q. The mathematician continues to define other logical situations to which he refers by special term. Thus ~q $\Rightarrow$ ~p is the contrapositive of p $\Rightarrow$ q and ~p $\Rightarrow$ q is the inverse of p $\Rightarrow$ q. The student of mathematics will recall this meaning of 'inverse' as being different from the one above.

In number algebra an inverse can exist only if an identity element exists.. For addition the identity element is zero. If a+b=o, then a and b are inverse of each other, i.e. a = -b, -b being the inverse of a, or b=-a, -a being the inverse of b. Observe that both a and b cannot be negative at the same time. In multiplication 1 is the identity element.

If ab=1, so the inverse of $a=\frac{1}{b}$, and the inverse of $b=\frac{1}{a}$.

We refer to the first part of this paper.

p: The integer a such that a>5;

q: The integer a such that $a^2>25$

Here we see the integer a is such that a>5, then $a^2>25$. So we write p $\Rightarrow$ q. This is true.

Again we refer to the first part of this paper.

p : The integer a is odd;

q: The integer a^2 is even.

Here we see if a is odd, then a^2 is even. But this is not true. This is summarized in the following table.

Table 8: Implications of propositions

p	q	p $\Rightarrow$ q
T	T	T
T	F	F
F	T	T
F	F	T

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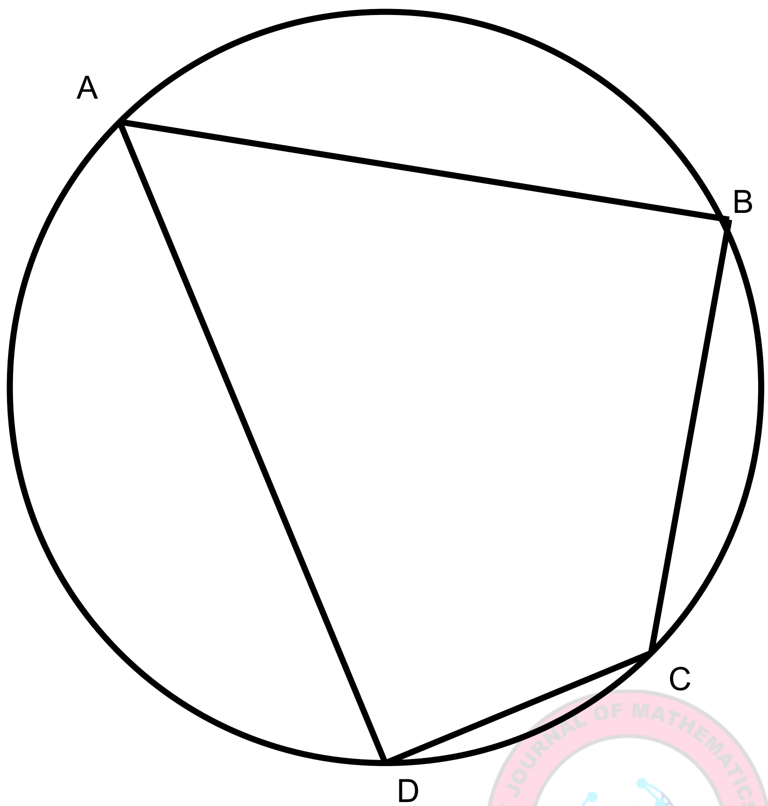


Figure 14: Cyclic quadrilateral

In the above diagram ABCD is a quadrilateral in the circle. This is known as a cyclic quadrilateral. It can be shown that the opposite angles of this quadrilateral are supplementary, i.e. <BAD + <DCB = 180°. This idea or theorem in Geometry is further mathematized as follows:

p: quadrilateral ABCD is cyclic
q: quadrilateral ABCD has opposite angles supplementary.

$p \Rightarrow q$: If quadrilateral ABCD is cyclic then its opposite angles are supplementary.
$\sim q \Rightarrow \sim p$: If the opposite angles in the quadrilateral ABCD are not supplementary, then the quadrilateral is not cyclic.

The following truth table summarizes and at the same time extends the implication between statements p, q, ~q, ~p.

Table 9: Implication of negative positions

p	q	$p \Rightarrow q$	~q	~p	$\sim q \Rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

From the above we see that the truth value for $p \Rightarrow q$ is the same for the truth value for $\sim q \Rightarrow \sim p$, the contrapositive of $p \Rightarrow q$. Thus $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent.

It is in this vein of reasoning the mathematician creates mathematics and provides elegant and powerful proofs of theorems whose truth is difficult to invalidate.

4.1 Switching Circuits

The laws of switching circuits are a direct counterpart of symbolic logic.

There are two basic circuits, viz. series and parallel. A combination of these basic circuits leads to complex circuits whose laws are the same as those of symbolic logic.

Fig. 15: Series connection

Here two switches are connected in series Fig. 15. Current can flow only when both p and q are on, i.e. closed. In symbolic logic this corresponds to $p \wedge q$.

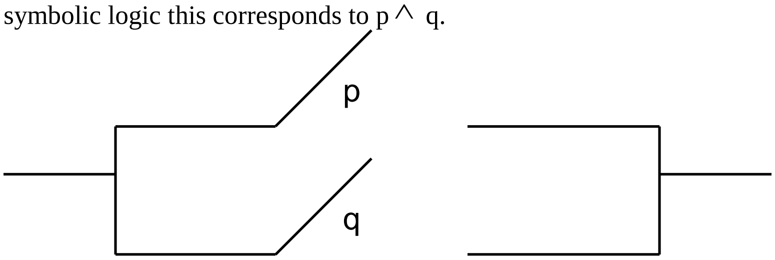


Fig. 16: Parallel connection

Current can flow through this circuit if either p or q is closed or both are closed. This corresponds to $p \vee q$ in logic. The following diagram (Fig. 17) shows an arrangement where one switch is closed, the other is open. And since this is a series circuit no current flows. This is expressed by $p \wedge p^1=0$, where p^1 represents the closed switch and 0 denotes no current flows.

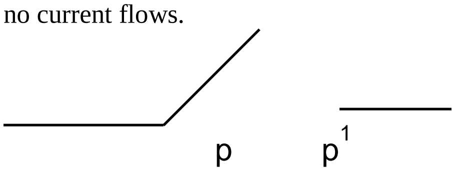


Fig. 17: Switches open and closed in series

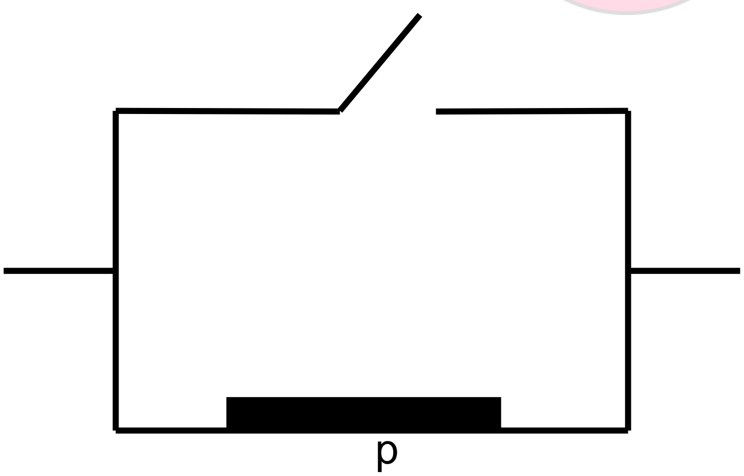
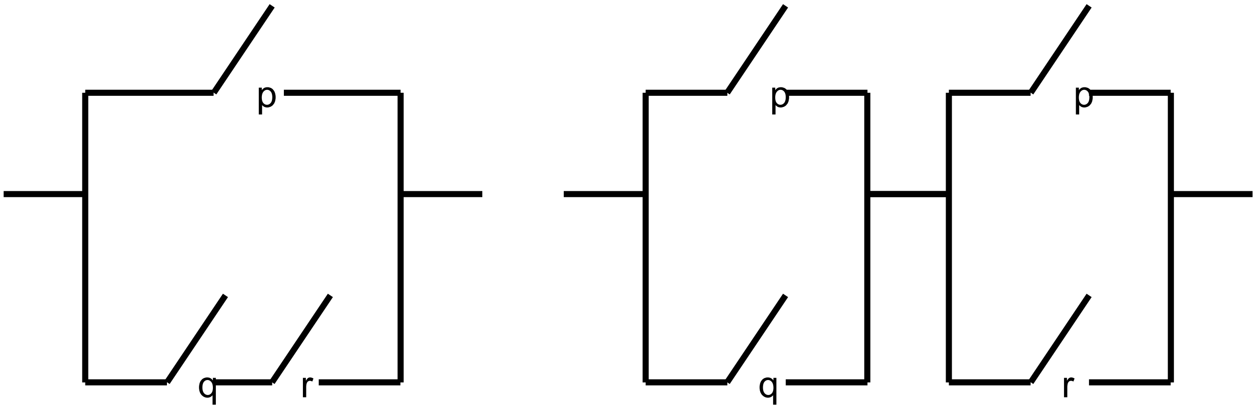


Fig. 18: Switches open and close in parallel

The above diagram is a parallel circuit with one switch, p^1, closed. Current must flow and the equation $p \vee p^1 = 1$ represents this, where 1 denotes current flow.

These ideas the mathematician uses to create a body of mathematical knowledge which can inform engineers when they prepare electrical networks e.g. the wiring of buildings or electronic equipment as T.Vs, radios, computers, etc.



(a) (b)

Fig. 19a & b: Parallel-series connections

In diagram (a) Fig. 19, if p is closed, current flows. If p is open, current flows if and only if both q and r are closed. And if all switches are closed current flows. The mathematician expresses this symbolicallu as

$$p \vee (q \vee r) = 1$$

In diagram (b) Fig 19, if p or q is closed and p or r is closed current flows.
This is shown by the equation

$$(p \vee q) \wedge (p \vee r) = 1$$

Equations (1) and (2) give the same results and the mathematician says that these equations are equivalent. Thus

$$p \vee (q \wedge r) = (p \vee q) \vee (p \vee r).$$

This is the distributive law, i.e. $\vee$ is distributed over $\wedge$. In number algebra if a, b, c are numbers, then

$$a(b+c) = ab + ac$$

Multiplication is distributed over addition. A similarity between switching circuits and number algebra is emerging. We now examine the following circuits

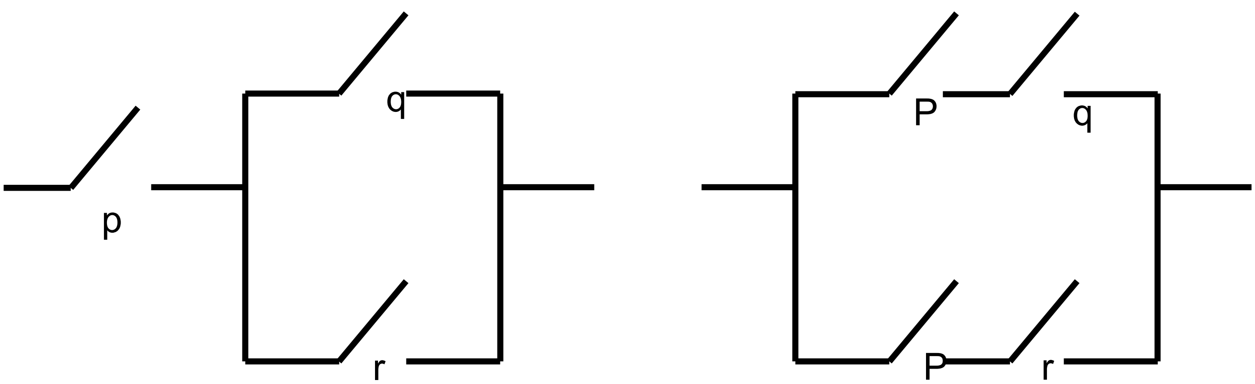


Fig. 20: Complex parallel series connections

If p is open no current flows through the circuits (c) Fig. 20 and (d) Fig. 20. In circuit (c) current flows if p is closed and either q or r or both are closed. The equation for this is

$$p \wedge (p \vee r) = 1$$

In circuit (d) if p and q and r are closed current flows. The equation for this is

$$(p \wedge q) \vee (p \wedge r) = 1$$

Equations (3) and (4) are equivalent. Thus

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

This is the distributive law, and $\wedge$ is distributed over $\vee$.
An examination of the following circuit Fig. 21 reveals that if p and q^1 are

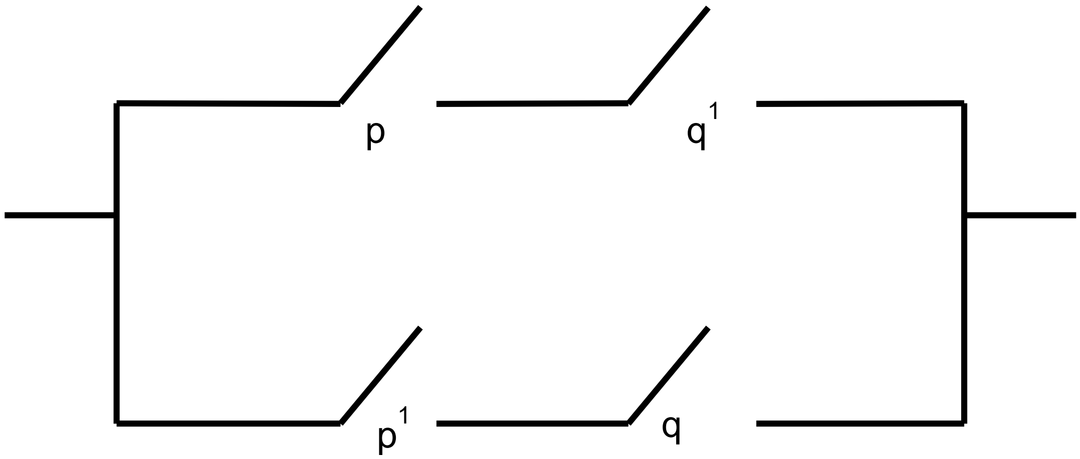


Fig. 21: Parallel-series connections

closed Again $(p^1 \wedge q)=1$ if p^1 and q are closed. To put these together we have the function $(p \wedge q^1) \vee (p^1 \wedge q)$ and this is 1 if either $(p \wedge q^1)=1$ or $(p^1 \wedge q)=1$. All cases are considered in the following table.

Table 9: Summarizes the functions of various circuits

p	q	p^1	q^1	$p \wedge q^1$	$p^1 \wedge q$	$(p \wedge q^1) \vee (q \wedge p^{1)}$
1	1	0	0	0	0	0
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	0	0	0

By considering the behavior of electricity and the function of switches, the mathematician, using symbols to represent switches and symbols to represent ‘and’ and ‘or’ has created a mathematical system whose laws are the same as those of symbolic logic. And the laws of symbolic logic are the same as those of set algebra. Thus set algebra, symbolic logic and switching circuits obey the same laws, in spite of the fact that they are derived from different ways of thinking to solve different kinds of problems. These laws, viz. commutative, distributive and associative, seem to have a unifying effect on set algebra, symbolic logic and switching circuits. Is there a body of mathematical knowledge or laws which can unify all mathematics as all the physical laws with one flourish of the pen?

5.0 THE CASE OF FINITE MATHEMATICS

What practical use a mathematician can make of the remainder when one number is divided by another e.g. $47 \div 6 = 7$ remainder 5? One response is that $7x6 + 5 = 47$, and to generalize this we have to say, P and Q are divisor and dividend respectfully, and R as remainder, Q $\div$ P=A+R. If R=O, then AP=Q, and we say that A and P are factors of Q. This is known as the remainder theorem and has uses in the solution of algebraic equations. This is perhaps as far as we can go with remainders in elementary finite mathematics. Let us look at this problem. It is now 7:00 a.m., and Air Caribbean leaves for London in 9 hours time. At what time does Air Caribbean leaves for London? A common sense approach to this problem can quickly furnish the correct answer. We are used to counting from 12 o’clock and we may break up 9 hours like this. From 7 o’clock to 12 o’clock is 5 hours. Four hours remain. Four hours from 12 o’clock brings the clock to 4 o’clock pm. And this answers the question above.

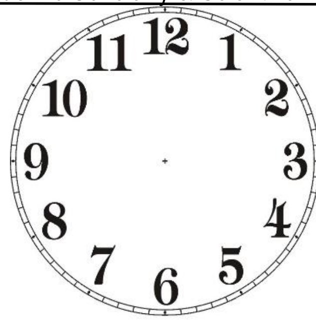


Fig. 22: Clock dial

The above reasoning is good mathematical reasoning. However, the mathematician looks for some pattern from which he can make generalizations and from these useful inferences and deductions. The mathematician examines the analogue clock in Fig. 22 and observes that counting is from 1 to 12, after which the counting is repeated from 1 to 12 i.e. there is no number beyond 12, so there will be repetition of counting from 1 to 12. This presents a finite system of counting, unlike and infinite system where counting goes on ad infinitum. Yet does a finite system present a fertile ground for the spawning of new and useful ideas?

The mathematician handles the departure time of Air Caribbean like this. We recall the problem. It is 7 o'clock. In 9 hours Air Caribbean Departs. What time will this be? He adds 7 to 9 i.e. 7 + 9=16. But this is not a number on the analog clock. The counting passes the number 12 and goes to 1 to 2 etc. This situation gives the mathematician a clue. He divides 16 by 12 and gets the remainder 4. This remainder answers the above question. The mathematician writes his reasoning like this.

7+9=4(modulo 12) also written as 7+4(mod 12). Here the addition is slightly different, and this is expected in a finite set, where addition can go on and on.

Here the remainder when one number is divided by another proves to be useful and is the focus of this new arithmetic called remainder arithmetic or more popularly as modular arithmetic. Let us look at the above problem the other way around. Air Caribbean departs at 4 p.m. Nine hours ago this departure was announced. What time was it then? The mathematician sees this as the equivalent of subtraction and so
4 – 9 = -5. This is interpreted as 5 hours before 12 o'clock. The mathematician adds 12 to -5 and gets 7 a.m., i.e.
-5 + 12 = 7
Let us try another example. It is now 11 a.m. In 42 hours what will be the time? Here it is that 11 + 42=53. Now divide 53 by 12, we have $\frac{53}{12}$ = 4 and remainder 5. The remainder tells us the time. It will be 5 p.m. This can be written thus:
11 + 42 = 5 (mod 12)

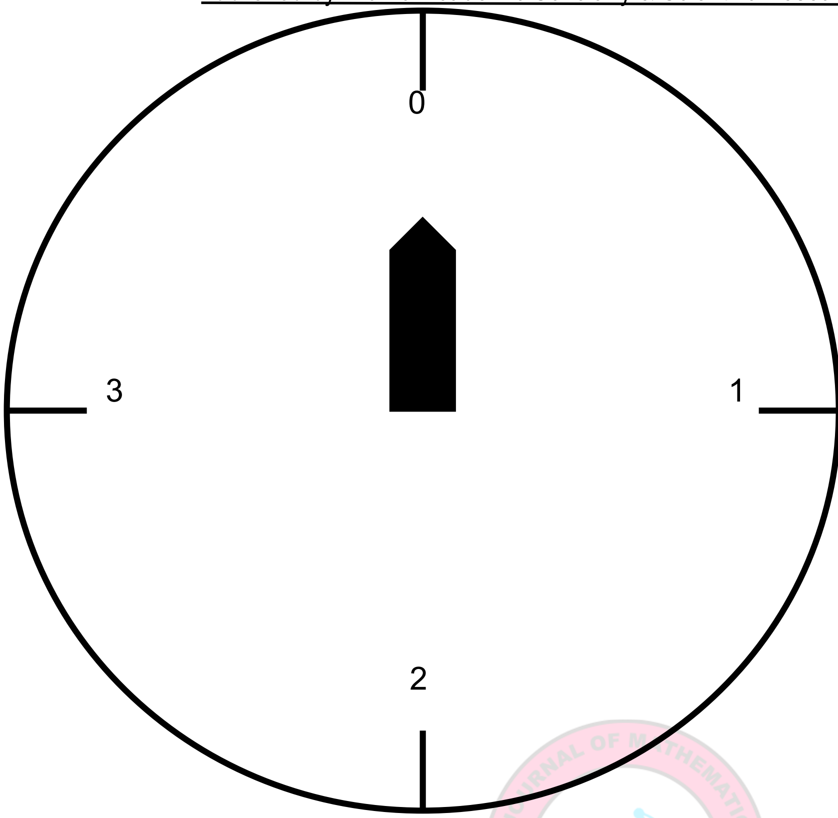


Fig.23: Control switch of fan

Fig. 23 shows the control of a fan. This is a rotary switch. The fan is off at 0, and at 1 the fan speed is lowest, at 2 the speed is increased and at 3 the speed is maximum.

The mathematician carefully observes the above rotary switch and notes that:

(1) There exists in this system a set of elements {0,1,2,3}.

(2) That an operation – rotate – is implied. He defines and symbolizes 'rotate' as $\circlearrowright$ which means 'rotate clockwise' and $\circlearrowleft$ which means 'rotate anticlockwise'.

$\circlearrowright$ and $\circlearrowleft$ are slightly different from ordinary addition and subtraction respectively. The mathematician applies the operation $\circlearrowright$ on the above elements and notes the result.

1 $\circlearrowright$ 2=3. This means we start from 1 and turn the switch 2 places. This brings us to 3. Of course, this is the maximum speed of the fan, and this is just like ordinary addition. Now we try this 2 $\circlearrowright$ 3=?. Is it 5? No! Five does not exist in this system. Then what is the answer? Let us do what $\circlearrowright$ says. From 2 rotate clockwise 3 places. We come to 1. Thus 2 $\circlearrowright$ 3=1. The result 1 is a member of the set of elements. It is interesting to note if we apply the

operations $\circlearrowleft$ to the above example or as a matter of fact to any example in this system e.g. 2 $\circlearrowleft$ 3=? Start from two and rotate anticlockwise 3 places. We come to 3. Thus 2 $\circlearrowleft$ 3=3. Let us try another example. 3 $\circlearrowright$ 3=? From the point 3 move clockwise 3 places. We come to 2. So 3 $\circlearrowright$ 3=2. The mathematician writes this like thus:3 $\circlearrowright$ 3=2(mod 4). This system is indeed an example of modular arithmetic where the sum of given elements in this system is divided by 4 and the remainder answers the question. Thus 3 $\circlearrowright$ 3 can be interpreted as (3+3) ÷ 4=2. The mathematician uses this idea and creates a table and from the results of the operation $\circlearrowright$ he makes the observation.

Table 10: Summary of the results of the operation $\circlearrowright$

$\circlearrowright$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	1

The mathematician notes that 1 $\circlearrowright$ 0=3 and 2 $\circlearrowright$ 1=3, 3 $\circlearrowright$ 1=0 and 1 $\circlearrowright$ 3=0, 2 $\circlearrowright$ 3=1 and 3 $\circlearrowright$ 2=1. He observes that the order of doing $\circlearrowright$ does not affect the results and so he generalizes the observation. If a and b are any elements in the system, q $\circlearrowright$ b=b $\circlearrowright$ a.

1. This is the commutative law and the operation $\circlearrowright$ is said to be commutative. Again (1 $\circlearrowright$ 2) $\circlearrowright$ 3=? The brackets tell us to do 1 $\circlearrowright$ 2 first, then use this result and perform $\circlearrowright$ on 3. So 1 $\circlearrowright$ 2=3, and 3 $\circlearrowright$ 3=2. Thus (1 $\circlearrowright$ 2) $\circlearrowright$ 3=2. Similarly we see 1 $\circlearrowright$ (2 $\circlearrowright$ 3)=1 $\circlearrowright$ 1=2. The mathematician notes that (1 $\circlearrowright$ 2) $\circlearrowright$ 3=1 $\circlearrowright$ (2 $\circlearrowright$ 3) and generalizes (a $\circlearrowright$ b) $\circlearrowright$ c=a $\circlearrowright$ (b $\circlearrowright$ c).
2. This is the associative law for the operation $\circlearrowright$.
3. In any row or column, no element in this system is repeated. The mathematician observes the existence of the element 0. 2 $\circlearrowright$ 0=2 and 0 $\circlearrowright$ 2=2. For any element in the system a $\circlearrowright$ 0=0 $\circlearrowright$ a is true. Zero is called the identity element under the operation $\circlearrowright$.
4. There exists in this system an identity element under $\circlearrowright$.

5. Table 10 is closed or self-contained, i.e. no other element besides the ones defined in this system appears. This system is closed under the operation $\circlearrowright$. We say closure is observed.
6. We see the operation $\circlearrowleft$ is possible under this system e.g. 1 $\circlearrowleft$ 3=?

From Table 10 we can read the result. We look along the bottom row labeled 3 and we come to the number 1. This is vertically below the number 2 in the top row. So 1 $\circlearrowleft$ 3=2. We read the table the opposite way we would read it when doing the operation $\circlearrowright$. So $\circlearrowleft$ is possible in this system. The mathematician says that for any element a, b, c in the system, a $\circlearrowleft$ b=c.

In modular arithmetic multiplication is slightly different from ordinary multiplication. Here is a table of multiplication.

Table 11: Summary of the results of multiplication

x	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

An examination of Table 11 shows that both the commutative and associative laws hold. In the second row the element 2 is repeated and 0 appears. This is not one of the original elements in the above table. Properties 3 and 5 do not exist under the operation x. Again the operation division under this system does not give a unique result because 2 x 1 = 2, so that 2 ÷ 2 may result in 1 or 3. We notice that 1 x 2 is the same as 2 x 1, i.e. 1 x 2 = 2 and 2 x 1 = 2. So in multiplication there is an identity element which is 1, but no inverse element, for in this system if a is an element and if there exists an inverse element, say, x, then a x x = 1. Then $x = \frac{1}{a}$. But $\frac{1}{a}$ is not an element of this system and therefore x does not exist. Therefore there is no inverse element in this system for multiplication. Under the operation $\circlearrowright$ we observe that : - (See Table 10)

1. Closure is observed, i.e. if a, b are elements in the system and a $\circlearrowright$ b=c, where c is also an element of the system.
2. The system in Table 10 shows that there is an element 0, such that a being an element in the set a $\circlearrowright$ 0 = 0 $\circlearrowright$ a = a. Zero is the identity element for the operation $\circlearrowright$.
3. There exists an element a, that for every a there is another element b such that b $\circlearrowright$ a = a $\circlearrowright$ b = 0, the identity element for $\circlearrowright$. Then a is said to be the inverse of b under the operation $\circlearrowright$.
4. The associative law is true under the operation $\circlearrowright$, for a, b, c, elements of the system a $\circlearrowright$ (b $\circlearrowright$ c) = (a $\circlearrowright$ b) $\circlearrowright$ c = a $\circlearrowright$ b $\circlearrowright$ c. If all four of the above features exist in a system, the mathematician defines that system as a group under the given operation. Under x (multiplication) the system does not form a group as all the above features do not exist.

Thus the Table 10 for arithmetic modulo 4 for the operation $\circlearrowright$ is a group. If a group has an addition property under the operation in question such that a * b = b * a are elements of the group, and * the operation, the mathematician says that this group is a commutative or Abelian group after the mathematician Abel.

We observe that under the system arithmetic modulo 4 the set of elements {0, 1, 2, 3} are considered. These numbers carry dual meanings. In the first instance they refer to points on the dial of the rotary switch, and secondly they refer to the spaces between these points. For example (2 $\circlearrowright$ 3) means we start at 2 and then move 3 spaces clockwise. We reach 1 which specifies a point. Mathematicians use this ambiguity to create mathematics which in many instances address as problems of the phenomenal world and in some instances the mathematics go into deep abstraction way and beyond the phenomenal world.

From the simple example of the rotary switch of a fan the mathematician weaves a complex system of mathematics called group theory which plays an important part in contemporary mathematics and science. Historically group theory was used in connection with the solution of equations. Now group appears not only in modular arithmetic, but in crystallography, in quantum mechanics, in geometry and in topology, in analysis and algebra, in physics and chemistry and even in biology, and may even in one theorem unify all science some day or in the future!

The set integers {… -3, -2, -1, 0, 1, 2, 3, …} form a group under addition as all the properties of a group hold. The same can be said of the even numbers, both positive and negative. The odd numbers under addition do not form a group as we see that 3 + 5 + 8. Three and 5 are odd numbers but 8 is even. Here the property of closure does not hold.

In geometry a symmetry of plane figures and also of solid shapes as well as reflection of points about defined axes in a plane form groups. It is interesting to note that the set S= {1, i, -1, -i} where $i = \sqrt{-1}$ form an Abelian group under multiplication. The following is a multiplication table. The symbol · stands for multiplication.

·	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

1. The table shows that all results of multiplication are elements of the set S. Closure is observed.

2. We examine $(1\cdot i)\cdot -1$ and $1\cdot(i\cdot -1)$ as an example.

$(1\cdot i)\cdot -1 = i \cdot -1 = -i$
$1\cdot(i\cdot -1) = 1\cdot -i = -i$
So that $(1\cdot i)\cdot -1 = 1\cdot(i\cdot -1)$. The associative property holds. And this can be shown to be true for all a, b, c €S.

3. The multiplicative identity does exist in S and this is 1. For all a € S we have 1·a = a·1 = 1.

4. From the table we see that all elements have inverses e.g. $1^{-1} = 1$, $i^{-1} = -i$ $(-1)^{-1} = (-i)^{-1} = i$

Thus the set S form a multiplicative group.
Here we see the mathematician observes the behavior of a non empty set under a given operation and those behaviors that are consistent he identifies, describes and names, and calls them properties or laws of the non empty set under the given operation. If other systems emerge, he examines them and if they possess the above four properties, they are classified as groups under a defined operation.

The question of √-1 is the subject of the next section of this paper.

6.0 FROM THE REAL TO THE UNREAL

We plot the function $y = x^2 - x - 6$ on graph paper (Fig. 24). We see the function cuts the axis at $x = 3$, $x = -2$. At these points $y = 0$. So we have in this way solved the quadratic equation $x^2 - x - 6 = 0$. We say the roots of this equation are 3 and -2, i.e. these are the values of x that will satisfy the above equation.

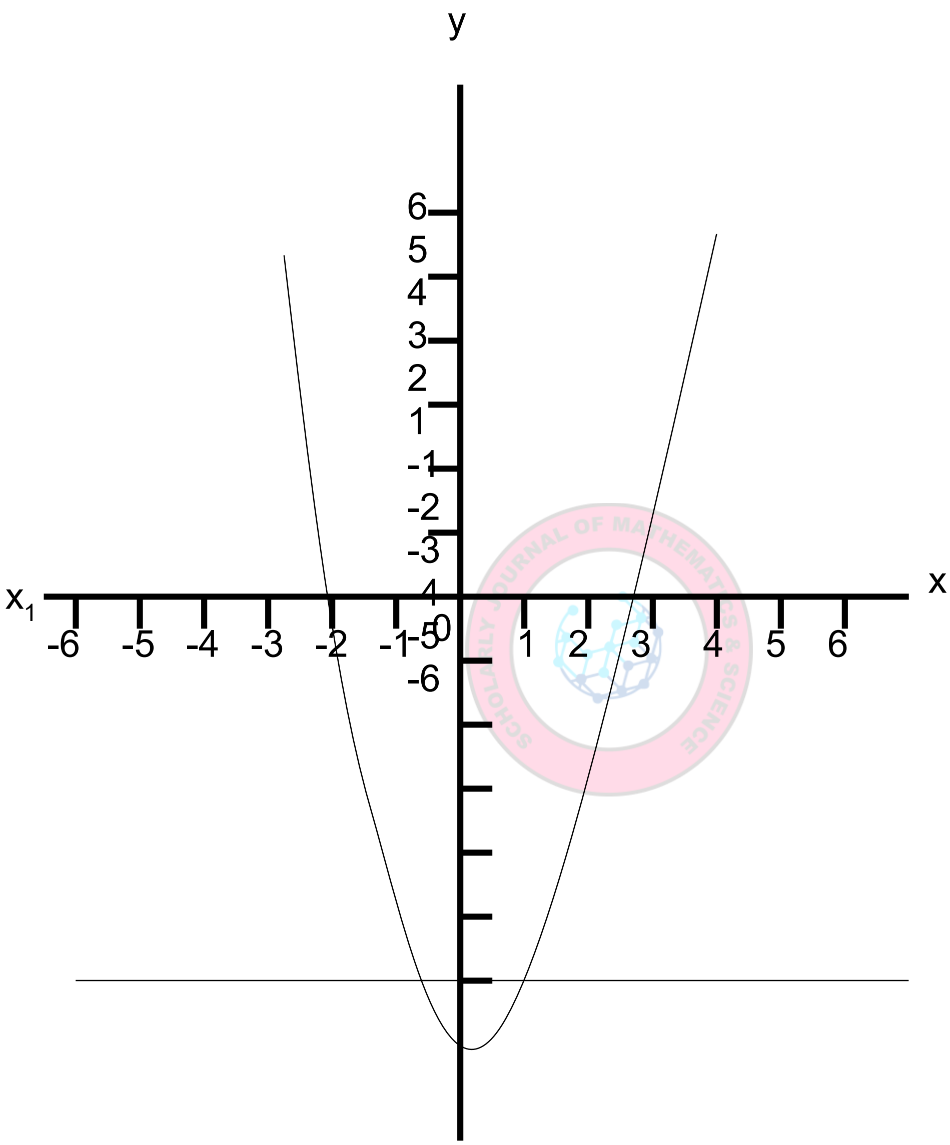


Fig. 24: Graph of y= x^2**-** x **-6**

Here is a quadratic equation: - x^2 + 1 = 0. The graph (Fig. 25) of the function y= x^2 = 1 shows that the function does not cut the x -axis, and therefore no value of x on the graph exist that will make y=0. An algebraic approach to the solution of y = x^2 + 1 is to assume that y = 0 so x^2 + 1 = 0.

Then x^2 = -1

∴ x = ± √-1 i.e. x is either +√-1 or -√-1. These values of x substituted in the above function make y = 0. But these numbers do not exist on the graph nor do they exist in the set lf real numbers. But why? Let’s see.

$+2x+2=+4$ and

$-2x-2=+4$

$\therefore \sqrt{+4} = +2$ or -2

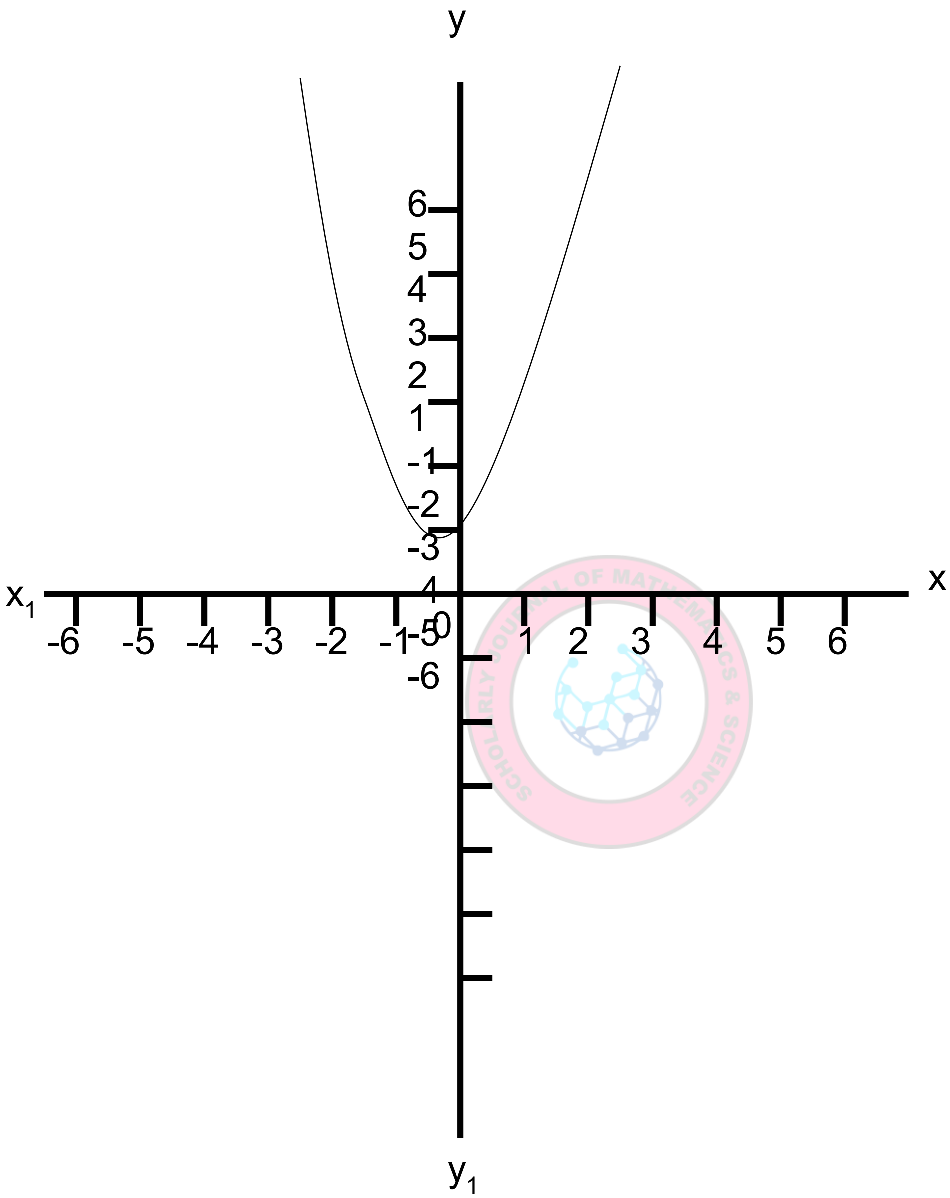


Fig. 25 : Graph of y = x^2 + 1

The square root of positive numbers do exist, nut the square root of negative numbers do not exist, i.e. there is no known real number that is the square root of a negative number, as we see above the square of a real number, whether positive or negative is always positive. This baffled mathematicians for some time: they were awe-struck. But mathematicians do not give up easily. This just called √-1 an unreal number or imaginary number. When √-1 is squared its results in a real number, -1. This made the mathematician suspicious that there is something promising about √-1. For convenience he writes i for √-1. But he wonders if i a number is at all. The mathematician observes the behavior of i when it is raised to positive powers. Thus

$i^2 = -1$ $\qquad$ $i^3 = -i$ $\qquad$ $i^4 = 1$

$i^5 = i$ $i^6 = -1$ $i^7 = -i$

$i^8 = 1$ $i^9 = i$ $i^{10} = -1$

$i^{11} = -i$ $i^{12} = 1$ $i^{13} = i$

This forms a sequence
-1, -i, 1, i, -1, -i, 1, -i, -1, -i, 1…, and (-1, -i, 1, i) repeats itself. It starts with an even power of i and ends in an odd power of i at every fourth term. We have seen before that the set {-1, -i, 1, i) forms a group under multiplication. This suggests that i raised to positive powers gives results that show the properties of real numbers in modular arithmetic under multiplication. This pleases the mathematician as i behaves like a number he is familiar with, i.e. a real number.

The solution of $x^2 + 4x + 13 = 0$ are $2 \pm 3i$ and $-2 - 3i$. These numbers have two parts, a real part, number -2, and an unreal imaginary part, +3i and -3i though +3 and -3 are unreal. The mathematician subjects the above numbers to the familiar operations (+, -, x, ÷) of algebra and finds that most of the properties e.g. commutative, associative etc. hold. He welcomes numbers of the form a + bi in the fold of real numbers with an adjustment which becomes necessary as a + bi is part real and pat imaginary. The real numbers are extended to include numbers like a +bi and are now called complex numbers denoted by C. Thus the real numbers belong to the set of complex numbers and are often referred to as complex numbers also, but without the imaginary part.

Can the complex number say, -2 + 3i, or for that matter any complex number a+bi be shown graphically? J. R. Argand)1768-1822) showed that this is possible. He called the x-axis of the Cartesian plane the real axis, points on which represent real numbers, and the y-axis the imaginary axis, points on which besides the origin represent imaginary numbers. The Cartesian plane in the above context was renamed the Granssian plane (after (Granss, 1777-1855) or simply the complex plane. The points -2 + 3i and -2 - 3i are shown on the plane (Fig. 26). We see that P_1 (-2 - 3i) is the image of P (-2 + 3i) by reflection about the x-axis or the real axis. Complex numbers have implications for transformation geometry and do in fact provide a powerful tool in interpreting problems in that area. They also provide neat proofs of a host of geometrical theorems.

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are two complex numbers, then they can be shown on the complex plane by, say, P_1 and P_2. See graph (Fig. 27). The diagram they form is called the Argand diagram. Now Z1 + Z2 is possible on this plane. Let $Z_1 = 5 + 2i$, and $Z_2 = 1 + 3i$, then $Z_1 + Z_2 = 6 + 5i$ and this is shown on the complex plane as P_3. The mathematician sees that this is analogous to vector addition as $Z_1 + Z_2$ is the same as $OP_1 + OP_2$ i.e. $Z_1 + Z_2$ obeys the parallelogram Law for the addition of vectors. Note that P3 generated from $Z_1 + Z_2$ forms the fourth vector of the parallelogram $OP_1P_3P_2$. This suggests that it may be convenient to associate $x + yi$ with a vector OP as complex numbers corresponds exactly to vector addition.

The work of Ganss has put complex numbers on a sound theoretical footing giving these numbers legitimacy in the mathematical domain. Complex numbers are used as powerful tools in physics, electromagnetism, thermodynamics etc., and of course in mathematics as an interpreter and organizer of recondite and difficult issues in mathematics.

Abraham de Moivre derived a very important relationship in trigonometry. He showed that $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$. If a student is required to find cos 12θ into more manageable parts, say, (6θ + 6θ) and use the expansion for cos (6θ + 6θ) which is cos 6θ cos 6θ – sin 6θ cos 6θ. Then 6θ is expressed as, say, 3θ + 3θ and the process is repeated. This is a student's nightmare. The use of de Moivre's theorem simplifies this; the only tedium is the expansion of $(\cos\theta + i\sin\theta)^{12}$ which is equated to $\cos 12\theta + i \sin 12\theta$ we will obtain cos 12θ sin 12θ in terms of powers of cos θ and sin θ in one shot. And these results are real. Thus us an example of how the unreal produces real results. The mathematician has already expected this, and the way he subjects i to the already established rules of algebra confirms his expectation. Complex numbers indeed simplifies the work in many areas of mathematics and science, and reduces the tedium in hefty calculations.

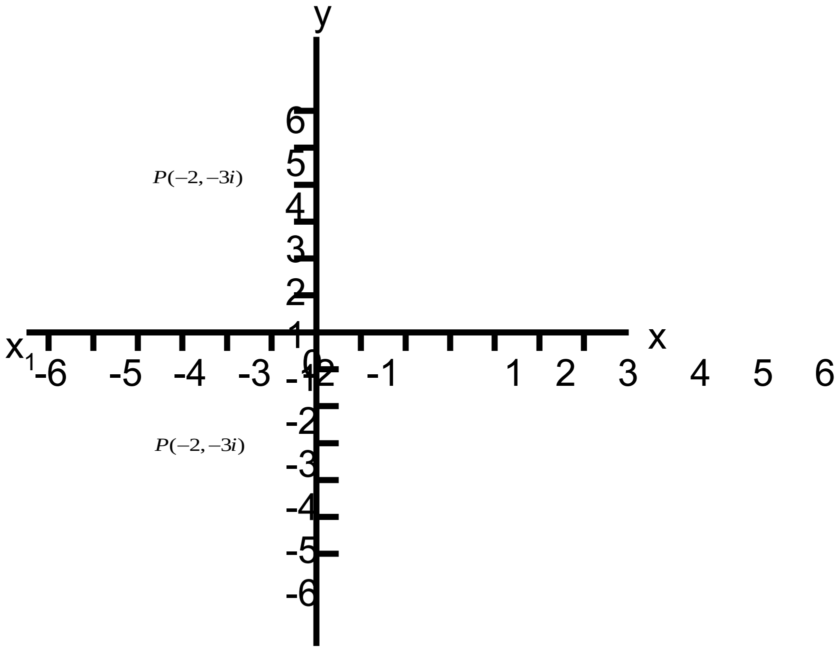


Fig. 26: Argand diagram

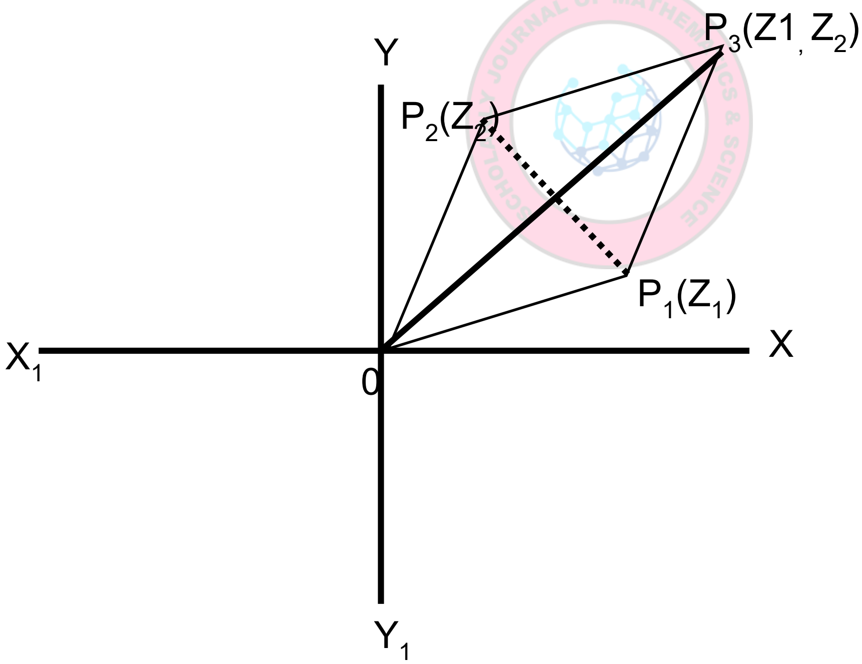


Fig. 27: Argand diagram

It is interesting to see how de Moivre derived the above trigonometrical relationship named after him. First we must recall the following already established trigonometrical identities.

(1) $\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta = \sin 2\theta$

(2) $\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta$

It is probable that he used the following line of reasoning.

$(\cos\theta + i\sin\theta)^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= \cos^2\theta + 2i \sin\theta \cos\theta + i^2\sin^2\theta = \cos^2\theta + 2i \sin\theta \cos\theta$
$-\sin^2\theta$ ($i^2 = -1$)
$= (\cos^2\theta - \sin^2\theta) + 2i \sin\theta \cos\theta$

From the above 2 identities, this reduces to $\cos 2\theta + i \sin 2\theta$… (1)
Again $(\cos\theta + i \sin\theta)^3 = (\cos\theta + i \sin\theta)^2(\cos\theta + i \sin\theta)$

$= (\cos 2\theta + i \sin 2\theta)(\cos 2\theta + i \sin\theta)$

$= \cos 2\theta\cos\theta + i \sin 2\theta\cos\theta + i \sin\theta\cos 2\theta + i^2 \sin 2\theta\sin\theta$

$= \cos 2\theta\cos\theta + i(\sin 2\theta\cos\theta + \sin\theta\cos 2\theta) - \sin 2\theta\sin\theta$

$= (\cos 2\theta\cos\theta - \sin 2\theta\sin\theta) + i(\sin 2\theta\cos\theta + \sin\theta\cos 2\theta)$

But $\cos 2\theta \cos\theta - \sin 2\theta\sin\theta = \cos(2\theta + \theta) = \sin 3\theta$ $\sin 2\theta\cos\theta = \sin\theta\cos 2\theta = \sin(s\theta + \theta) = \sin 3\theta$.
Then the above expression can be reduced to $\cos 3\theta$ to $i \sin 3\theta$… (2)
By following this procedure it can be shown that $(\cos\theta + i \sin\theta)^4 = \cos 4\theta + i \sin 4\theta$ and $(\cos\theta + i \sin\theta)^5 = \cos 5\theta + i \sin 5\theta$ and so on.
To generalize we have the famous theorem of de Moivre's.

$(\cos + i \sin\theta)n = \cos n\theta + i \sin n\theta$.

We see the usefulness of i which when squared reduces to the real number -1. This greatly simplifies the expansion and gives a neat and useful identity. If i is left out from the above identity then the expansion of $(\cos\theta + \sin\theta)^n$ by the binominal theorem leaves a result that cannot be simplified – a ersult that may not be useful.

7.0 MATHEMATICS AS A UNIFYING AGENT

To seek the hidden connections that underlie the unity of all things is to seek that knowledge through which all knowledge becomes known. Two or more areas in mathematics that appear to be separate and unrelated are indeed forms of one and the same thing. Arithmetic is the foundation of Algebra which as first seems distinct from Arithmetic, but in fact Algebra is Arithmetic generalized. Geometry lays the foundation for Trigonometry which has developed into a sophisticated discipline in its own right. Modern trigonometry looks like Algebra, and illustrations in trigonometry uses geometrical shapes, and to be more precise geometrical concepts, something quite different and distinct from the counting that is the foundation of Arithmetic! What then are the hidden connections that unify Arithmetic, Algebra, Geometry and Trigonometry?

Look at the following equations.

$y = mx + c$ (1)
$x^2 = y^2 = r^2$ (2)
$x^2/a^2 + y^2/b^2 = 1$ (3)
$y^2 = 4ax$ (4)
$(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (5)
$x^3 + y^3 = 3axy$ (6)
$(x^2 + y^2 - ax)2 = b^2(x^2 + y^2)$ (7)

They are indeed algebraic equations; they represent geometrical shapes! It was discussed earlier in this paper that a system of coordinates is used to define geometrical concepts in terms of algebraic concepts. Algebra submits very well to the laws and operations of Arithmetic and, therefore, is a more versatile mathematical tool than the rigid Euclidean geometry. It is Algebra that gives Euclidean geometry a dynamic life carrying it beyond the confines of a two dimensional plane to three or more dimensions with vigor and with ease. The concept of vectors is illustrated by Geometry and interpreted by Algebra. Non-plane geometrics are best interpreted by Algebra. This interpretation by Algebra gave the impetus to the growth of transformation geometry which uses a form of transformational operators called tenors which are really generalized vectors that are used to describe space not only in two three dimensions but in any number of dimensions. Geometrical illustrations in more than three dimensions are difficult or impossible – but

Algebra does the work! And it is the versatility of Algebra that helped Einstein to develop the theory of Relativity. It has been found by Charles and Gay-Lussac that the volume of a given gas or for that matter any gas under constant pressure varies with temperature, i.e. an increase in temperature results in an increase in the volume of gas, and a decrease in temperature results in a decrease in the volume of gas. The quantitative relationship between temperature and volume of gas turns out to be remarkably consistent. This relationship can be shown on the Cartesian plane as a geometrical line.

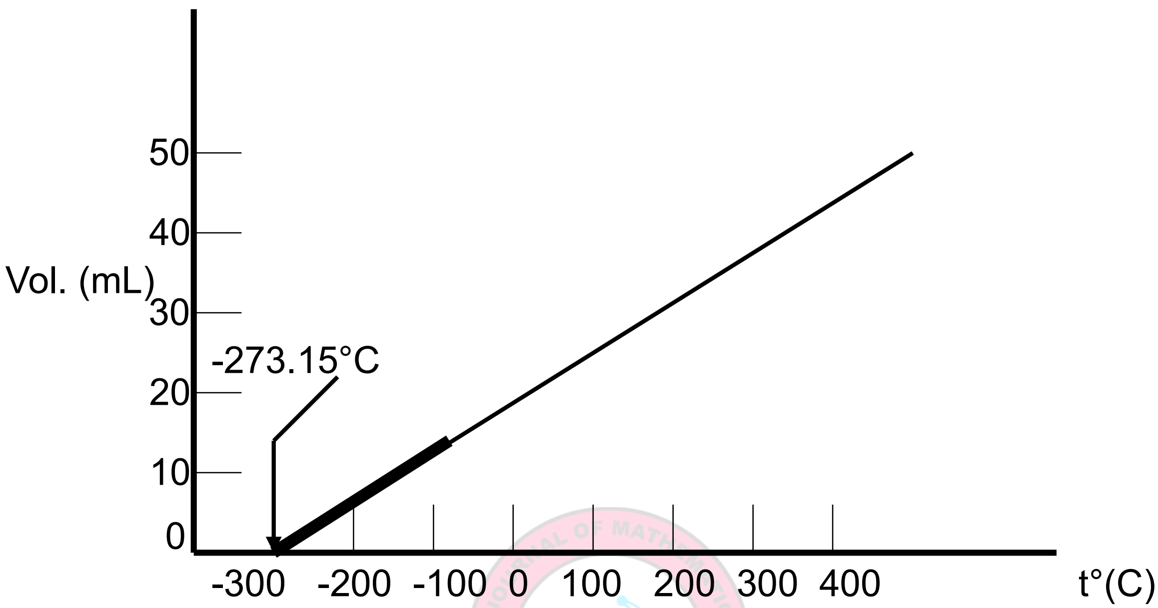


Fig. 28: Graph of a gas showing relationship between temperatue and volume, pressure being constant

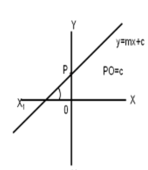
Temperatures of gas are plotted against volumes and this results in a straight line in the black (Fig. 28). The volume of a gas can be measured only over a limited range of temperature because all gases condense to form liquids at low temperatures. The line in black represents the results of laboratory experiments. At low temperatures when gases condense to form liquid no readings involving temperatures – volume relationship can be obtained. However, if this line is produced (shown in red) it intersects the temperature axis at a point -273.15ºC. This is called absolute zero on the temperature scale. Such temperature has not been achieved in the laboratory. It is worth noting that an absolute zero, the volume of gas is zero! At least theoretically this is so, but in reality is it so?

The above example shows the power of algebraic geometry in making prediction from observable trends. By quantifying this trend a relationship between temperature and volume can be made. Thus V = KT, where V is the volume of the gas and T the temperature, K constant. And this is Algebra – Algebra from Geometry. This has greater power to make predictions about the volume-temperature relationship of a gas than geometry can make. It is this power Algebra that gives scientists the tool to investigate and interpret natural phenomena. In the area of statistics, Algebra is the first organon and Geometry a means to present data and show trends visually.

What then is the common thread that binds Algebra and geometry? Are Algebra and Geometry ultimately one and the same things governed by some general principle? We now returned to the Algebraic equations stated earlier. The equations plotted on the Cartesian plane generate some surprising beautiful geometrical forms, a veritable gold mine for the creative artist and craftsman.

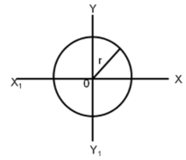
Algebraic Equations | **Geometrical representations**

1. $y = mx + c$ PO = C; $m = \tan \theta$



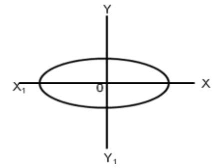
Straight line

2. $x^2 + y^2 = r^2$



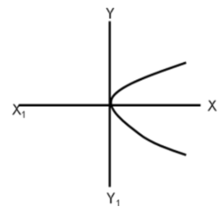
Circle

3. $x^2/a^2 + y^2/b^2 = 1$



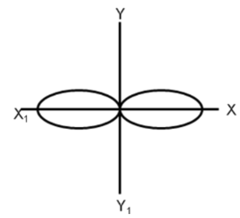
Ellipse

4. $y^2 = 4ax$



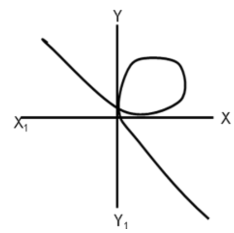
Parabola

5. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$



Lemniscate of Bernoulli

6. $x^3 + y^3 = 3axy$



Folium of Descartes

7. $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$

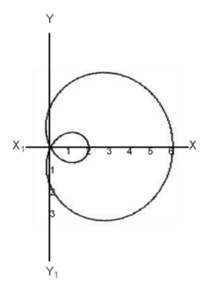


Fig. 29: Geometrical interpretation of Algebraic equations

The annexation of Geometry by Algebra gives mathematicians, scientists and statisticians tremendous power to summarize, present, analyze, and interpret natural phenomena, social phenomena and the phenomenon of human and animal behavior. In a large number of case prediction about the phenomena can be made e.g. by the use of Geometry and Algebra eclipses of the sun and moon had been predicted with great accuracy centuries ago. In Game theory developed by Von Newman uses Algebra to devise intricate laws of strategy to combat a shifty adversary with ease and economy. Geometry steps in to illustrate these laws and strategies. Geometry and Algebra seem to complement each other unifying mathematical thinking. From time to time many sequences and series have been discovered. A good many show similarities that at first glance might be considered an oddity or an unimportant curiosity. But on closer look they reveal a common underlying pattern. A binomial is an algebraic expression with two terms, e.g. 1 x x, a + b, y –z etc. a monomial is an expression with one term e.g. a x and a polynomial an expression with more than three terms, a three-term expression being called trinomial. Here we are interested in the expressions of binomials. Take the binomial (1- x) and expand it to various powers. This is what the mathematician does.

$(1 + x)^1 = 1 + x$
$(1 + x)^2 = 1 + 2x + x^2$
$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$
$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

The mathematician looks at the coefficient of x in each expansion and arranges them thus, including the constant 1 which appears fist.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

This arrangement is called Pascal's triangle after the mathematician Pascal. By carefully observing the above arrangement one can make a prediction about what the coefficient of the expansion of $(1 + x)^5$ would be.

In trigonometry we find that

$$\tan(A + b) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad (1)$$

Is there some pattern inherent in this equation? At first it might not appear so. Let us see what the mathematician does. He puts B = A.
Then the above becomes

$$\tan(A + A) = \frac{\tan A + \tan B}{1 - \tan A \tan A}$$ i.e.

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad (2)$$

For ease and economy the mathematician writes tan a = t. So equation (2) becomes

$$\tan 2A = \frac{2t}{1 - t^2} \ldots \quad (3)$$

By putting B=2A, tan 3A can be found and by putting B = 3A. tan 4A can be found. The mathematician puts these equations together and looks for a pattern. Thus

tan A = t

$$\tan 2A = \frac{2t}{1-t^2}$$

$$\tan 3\frac{3t-t^3}{1-3t^2}$$

$$Tan4a = \frac{4t-4t^3}{1-6t^2+t^4}$$

In the expansion of $(1 + x)^4$ the numbers 1, 4, 6, 4, 1 appear. In the expression of tan 4A the numbers without their signs appear like this.

4 4
1 6 1

But those are the same numbers in the expansion of $(1 + x)^{4.}$ There is sufficient evidence of a common underlying pattern in both expansions.

This requires further investigation which hopefully may uncover a subtle unifying principle which can explain the similarities between Trigonometry and Algebra and Geometry. However, it must be pointed out that at the onset of this argument the mathematician represents the trigonometrical ratio tan A by the algebraic symbol t and subjects to the laws of algebra. This produces useful results with ease and economy, and at the same time produces the same number patterns as seen in the binomial expansion and in the expansion of tan 4A. Is the representation of an idea by another idea the germ of an unifying principle?

An interesting phenomenon emerges from the idea of Compound Interest. Let us see how the mathematician handles this and makes use of it in ways that generate surprising and yet useful results.

Let P be the sum of money borrowed and r the rate per cent per annum. The interest at the end of the first year P x $\frac{r}{100}$ or $\frac{\Pr}{100}$. At the beginning of the second year the principal is P + $\frac{\Pr}{100}$, i.e. P(1 + $\frac{r}{100}$). The amount at the end of the first year is P x $\frac{r}{100}$ or $\frac{\Pr}{100}$. At the beginning of the second year the principal is P + $\frac{\Pr}{100}$, i.e. P (1 + $\frac{r}{100}$). The amount at the end of the first year is the principal for the second year, and the amount at the end of the second year is the principal for the third year and so on. This can be shown as follows:

Year	**Principal**	**Rate**	**Interest**	**Amount**
1	P	r	$\frac{\Pr}{100}$	$P+\frac{\Pr}{100}=P(1+\frac{r}{100})\frac{r}{100}$
2	$P(1+\frac{r}{100})$	r	$P(1+\frac{r}{100})\frac{r}{100}$	$P(1+\frac{r}{100})+P(1+\frac{r}{100})\frac{r}{100}=P(1+\frac{r}{100})^2$

By the same working as above, we get

Amount at the end of the 3rd year is P(1 + $\frac{r}{100}$)^3

Amount at the end of the 4th year is P(1 + $\frac{r}{100}$)^4

Amount at the end of the hth year is P(1 + $\frac{r}{100}$)^h

What will be above formula look like of the interest is added each half year? Then we will have:

Amount at the end of 1st half year is P(1 + $\frac{r}{2}$ x 100) because the rate is half of that for the whole year.

Amount at the end of the 1st year = P(1 + $\frac{r}{2}$ x 100)^2

Amount at the end of the 2nd year = P(1 + $\frac{r}{2}$ x 100)^4

Amount at the end of the nth year = P(1 + $\frac{r}{2}$ x 100)^{2h}

If the interest is added 4 times a year; then

Amount at the end of the last year = P(1 + $\frac{r}{4}$ x 100)^4

Amount at the end of the hth year = P(1 + $\frac{r}{4}$ x 100)^{4h}

If the interest is added 12 times a year then the amount at the end of the hth year is P(1 + $\frac{r}{12}$ x 100) ^{12h} and if the interest is added, say, m times a year the amount is P(1 + $\frac{r}{m}$ x 100)^{mh}

The above line of reasoning is a typical ay the mathematician uses to explore a given situation and if a pattern emerges he generalizes it. For the sake of ease of computation and economy the mathematician substitutes $\frac{1}{n}$ for $\frac{3}{100m}$ in the above formula which can now be written as:

Amount = p(1 + $\frac{1}{n}$)^{mh} …

Now from $\frac{r}{100m} = \frac{1}{n}, m = \frac{rn}{100}$

Replacing m in (1) by $\frac{rn}{100}$, we get

A = P(1 + $\frac{1}{n}$) $^{rnh/100}$, where A is the amount of after h years.

The above formula can be written as A = P{(1 + $\frac{1}{n}$)^n} $^{rnh/100}$. The mathematician calls (1 + $\frac{1}{n}$)^n the growth factor. It interest is added on to the principal at indefinitely small intervals, the growth of the principal may be considered to be continuous, and the amount reached will be the limit of $\{(1+\frac{1}{n})^n\}^{rnh/100}$ as n becomes infinitely large. Let us see how the mathematician does this. He uses the well known Binomial theorem to expand $(1+\frac{1}{n})^n$. Thus:

$$(1+1/n)^n = {}^nC_0(1/n)^0 + {}^nC_1(1/n)^1 + nC2(1/n)2 + nC3(1/n)3 + nC4(1/n)4 + \ldots$$

Thus , we have

$$(1+\frac{1}{n})^n = 1+\frac{n}{1!}(\frac{1}{n})^2 + \frac{n(n-1)}{2!}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{1}{n})^3 + \frac{n(n-1)(n-2)(n-3)}{4!}(\frac{1}{n})^{4} + \ldots$$

$$= 1 + \frac{1}{1!} + \frac{(1-\frac{1}{n})}{2!} + \frac{(1-\frac{3}{n}+\frac{2}{n^2})}{2!} + \frac{(1-\frac{6}{n}+\frac{11}{n^2}-\frac{6}{n^3})}{4!} + \ldots$$

We have already seen in this paper that as n→∞, i.e. n becomes infinitely large, expressions of the form $\frac{c}{n}$ become c/∞ and the value of this approaches 0. So the limited value of $(1+\frac{1}{n})^n$ as n→∞, written as

$$\text{Lt}\ (1+\frac{1}{n})^n = 1+\frac{1}{1!}+\frac{(1-\frac{1}{\infty})}{2!}+\frac{(1-\frac{3}{\infty}+\frac{2}{\infty})}{3!}+\frac{(1-\frac{6}{\infty}+\frac{11}{\infty}-\frac{6}{\infty})}{4!}+\ldots$$

$$= 1+\frac{1}{1!}+\frac{(1-0)}{2!}+\frac{(1-0+0)}{3!}+\frac{(1-0+0-0)}{4!}+\ldots$$

$$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

Thus the limited of $(1+\frac{1}{n})^n$ as n→∞ is an infinite series whose value has been calculated to be 2.7182 to 4 significant figures. This is denoted by e, so that e= $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots = 2.7182$.

The mathematician has found that e represents the natural law of organic growth and change. The mathematical expressions in many areas of physics, chemistry and engineering involve functions in which the variation is proportional to the function themselves, where e plays a key role, and e is used as the base of logarithm called natural logarithm which is very useful in computational problems involving circular and hyperbolic function (Trigonometry) statistics and probability and a host of other mathematical problems.

It can be easily shown that

$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots$ by the binomial expression.

The mathematician knows much about i (√-1) and he is curious to see how the function e^x behaves when x is replaced by i. He experiments. SO

$$e^x = 1+\frac{i}{1!}+\frac{i^2}{2!}+\frac{i^3}{3!}+\frac{i^4}{4!}+\frac{i^5}{5!}+\frac{i^6}{6!}+\ldots$$

$$= 1+\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}\ldots+i(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots)$$

after replacing even power of i^2 by +1 and odd powers of I greater than 1 by -1.

Expanding $e^{i\theta}$ where θ is an angle, we have from the above expansion

$$e^{i\theta} = (1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\ldots)+i(\frac{\theta}{1!}-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\ldots)$$

Maclaurin's expansions show that $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\ldots$

and $\sin\theta = \frac{\theta}{1!}-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{0^7}{7!}+\ldots$

Therefore $e^{i\theta} = \cos\theta + i\sin\theta$. This is due to the mathematician Euler. Replacing θ by π (180°) we get $e^{i\pi} + 1 = 0 \cos \pi + i \sin \pi$. But cos π = -1 and sin π = 0. Then the above becomes

$e^{i\pi} = -1$ or $e^{i\pi} +1 = 0$. This summarizes and unifies the whole analysis in one shot!

8.0 BEYOND EUCLID

Euclid based his geometry on the flat plane that extends to infinity in all directions. A point has position on the plane but no magnitude or size. This is difficult to conceptualize. Then too, a position can be defined only with respect to some frame of reference. There is none on the Euclidean plane. Is a point on the plane an abstract entity that resides in our imagination? Subjecting this imaginary point to the laws of reasoning, a coherent body of geometry has been built. If the point moves in one direction it describes a straight line. Then a line will have length but no thickness. Can such an entity as a line exist in the physical world? If a line moves on the plane at right angle to its length, it will describe a plane on the Euclidean plane. This plane is still Euclidean. This plane will have length and breadth – a two – dimensional figure – but no thickness. If this plane moves at right angle to its length and breadth it describes a solid shape in space, cube or cuboids which can be represented physically as this is in three dimensions. Is this so because we live in a three dimensional world? Applying the laws of reasoning to purely abstract as imaginary entities the mathematician builds a body of mathematics, both useful or potentially useful and sometimes speculative.

Lines on the flat plane (Euclidean) are said to be parallel if they do not meet no matter how far they are produced on the plane. Through two points one and only one straight line can be drawn parallel to the given line. These and other assertions are so obvious that they need no proof to establish their truth. They are called axioms. Mathematicians use these truths to establish the truth or other given mathematical assertions. In so doing they build a body of mathematical knowledge. It is interesting to note that this body of knowledge is developed from pre-established modes of thinking (deductive) that is not dependent on geometrical diagrams. On the Euclidean plane parallel lines do not meet, the sum of the angles of a triangle is always 180º, the size of the triangle does not matter, and Pythagoras Theorem holds. Again, if a straight line is perpendicular to a given line and another straight line is also perpendicular to the same given line, then these two lines are parallel. No need to prove this! Here is a diagram (Fig. 30) of the Euclidean plane. The arrows show that the plane extends to infinity in all directions. This plane can be best shown by an ever expanding circle.

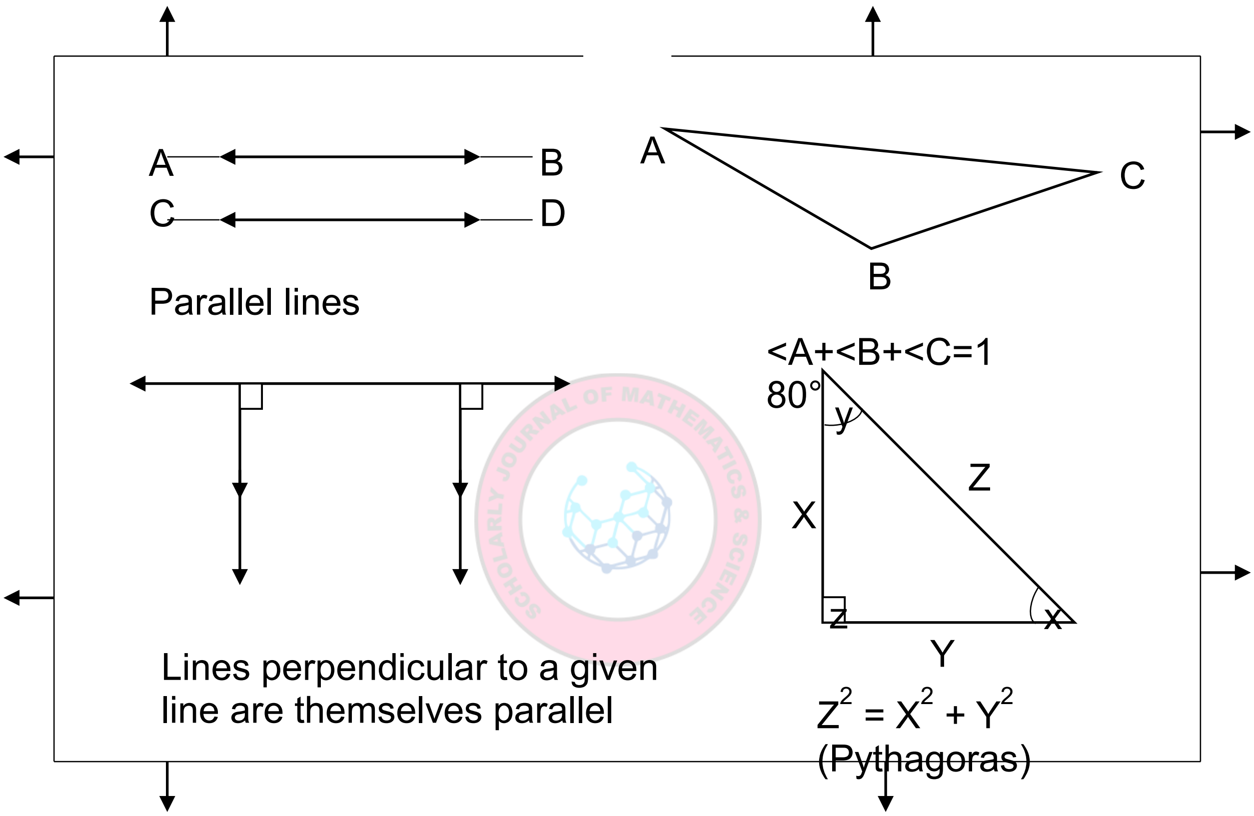


Fig. 30: The Euclidean plane

8.1 Moving On

During the nineteenth century mathematicians started to have doubts about Euclid's axiom which states that through a given point one and only one straight line can be drawn parallel to a given line. The published works of Labachewky, Bolyai and Riemann seriously questioned the axiom above. And particularly the woks of Riemann which shows that there are no such things as parallel lines. This jolted the mathematical world into furious debate, yet the works of Riemann laid the mathematical foundation for Einstein's theory of Relativity!

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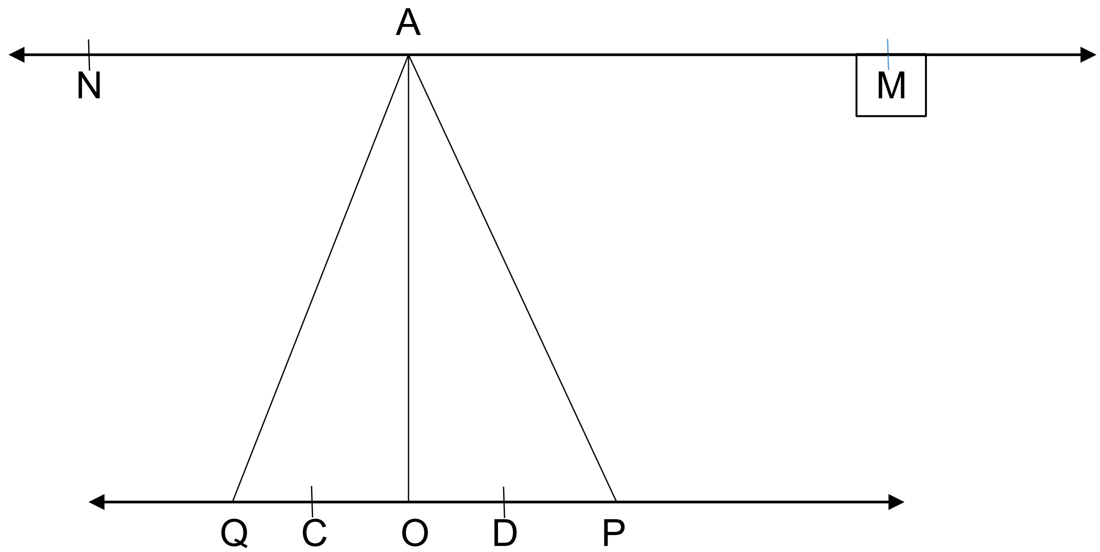

Fig. 31: Parallel line of axiom challenged

CD is a straight line, Fig 31. Through a given point A one and only one straight line can be drawn parallel to CD. This axiom has been challenged during the nineteenth century. The mathematician reasons like this.

P and Q are points on either side of O. P moves to the right and Q to the left. Lines AP and AQ swing towards the lien NM. As P and Q keep on moving the lines AP and AQ swing closer and closer to NM. When P and Q ae infinitely far from O, the lines AP and AQ are indefinitely close to NM but will never reach it. Euclid is right once the plan containing the lines shown in the above diagram is flat and unbounded.

As P and Q keep moving farther and farther away from O, the angle QAP gets larger and large and approaches the value 180º, but will not actually reach this value no matter how far P and Q move away from O. We say that the limiting value of the angle QAP is 180º. If P and Q have moved away from O at such a distance that 180º - LQAP = δ, where δ is less than a billionth of a degree, it will be hard to distinguish <QAP from <NAM. At the limiting value of <QAP, P and Q will rest on the lien NM, and PQ will become another line through A parallel to CD. But then there are an infinite number of lines that will do this. This contradicts Euclid, but we have seen before that this idea, though a paradox, is the genesis of the most powerful mathematical tool – the infinitesimal calculus. So there is an infinite number of lines passing through A parallel to CD, but all these lines coincide with the line NM and become one with it! There is a wry smile on Euclid's face.

The Mathematician Rieman argues as follows. The point P moves a finite distance, say r, from O to the right that OP intersects AM. This is NM meets CD at a distance r to the right of O. Again Q moves a distance r to the left of O so that OQ intersects AN. That is NM meets CD at a distance r to the left of O. Now two lines can intersect at two points seemingly. Are these two points different? The point obtained by going a distance of r to the right of O must be the same point by going a distance of r to the left of O. The line CD behaves like a circle. But this point lies on NM. Is this line NM also a circle? This cannot happen on the Euclidean plane. Riemannian geometry is based on a plane that is the surface if a sphere. This geometry gives a true interpretation of our universe than the geometry of Euclid.

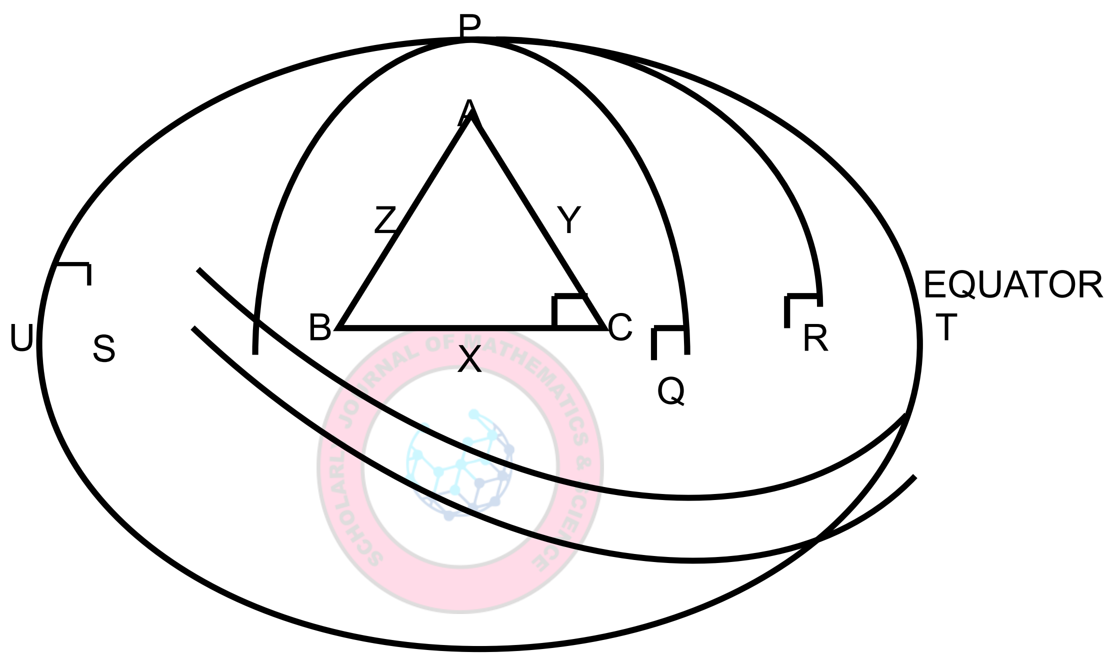

Fig. 32: The earth

The above figure 32 represents the earth. The equator is shown in read and P represents the location of one of the two poles. The lines PQ and PR are both perpendicular to the equator and therefore according to Euclid they are parallel. But if they are produced they meet at a finite distance at P. Therefore they are not parallel. If they are produced they meet again at the other pole and if they are produced further if they meet the equator in the points Q and R. These lines are not straight; they are circles. The theorems and axioms on parallel lines on the Euclidean plane do not hold on a spherical plane which is finite, i.e. the question of infinity does not arise and all distances on the sphere are finite.

The line UT is parallel to the equator and if produced in the direction $\overrightarrow{UT}$ it will meet itself at U, thus forming a circle parallel to the greater circle called the equator. It can be shown that all lines, parallel to the equator are themselves parallel to one another. Lines of Latitude are parallel.

The sum of the angles on a triangle on a sphere is not fixed. This sum varies according to the size of the triangle. The greater the area of the triangle, the greater the sum of its angles. In the above Figure 32 the angles of the triangle PRS are each 90º and therefore the sum of its angles is 270º. Again in the triangle PQ the angles at R and Q are each 90º and the angle at P is less than 90º. Therefore the sum of the angles of triangle PRQ is less than 270º. But then the area of triangle PRQ is less than the area of triangle PRS.

In the triangle ABC shown on the same figure above the angle at C is 90º and the sides of the triangle are all equal. It is clear that Pythagoras Theorem does not hold for spherical triangles. If the radius of the sphere is taken as the unit of measure the formula for the spherical triangle corresponding to Pythagoras is cos z = cos x cos y, where x, y, and z are the lengths of the sides of triangle ABC, z being the length of the hypotenuse. The mathematician continues his argument like this. If triangle ABC is very small its area will be small and as the triangle gets smaller and smaller, the sum of its angles approach 180º and its sides become less and less curved and starts taking the shape of a Euclid triangle.

There is a series for cosine, viz.,

$$\cos x = [1\frac{x^2}{x} - \frac{x^4}{24} - \frac{x^6}{720} + ...]$$

As x is small, x^4 and higher power can be neglected. Thus

$$(\cos x = 1 - \frac{x^2}{2})$$

$$(\cos y = 1 - \frac{y^2}{2})$$

$$(\cos z = 1 - \frac{z^2}{2})$$

Substituting these values of the of the cosines in cos z = cos x cos y, we have

$$1 - \frac{z^2}{2} = (1 - \frac{x^2}{2})(1 - \frac{y^2}{2})$$

This reduces to $\frac{z^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} - \frac{1}{4}x^2y^2$. As x and y are very small, $\frac{1}{4}x^2y^2$ can be neglected and the above becomes $\frac{z^2}{2} = \frac{x^2}{2} + \frac{y^2}{2}$ and removing the fractional part of each term, the equation reduces to $z^2 = x^2 + y^2$. This is the familiar form of Pythagoras Theorem. And this is expected as small parts of a sphere resemble small parts of a flat plan.

How different is Euclid's geometry from spherical geometry? Not much as most theorems of Euclid which do not depend on the idea of parallel lines hold true on the sphere. On the sphere the theorems on congruency of triangles hold, base angles of isosceles triangles are equal and all points on the bisector of, say, line AB are equidistant from A and B. The Theorem of Pythagoras and the sum of the angles of triangles (which vary according to the area of the triangle) do not hold.

9.0 TOPOLOGY, THE GEOMETRY OF DISTORTION

Over the last three hundred years many kinds of geometries have been developed. One of these is Topology. Topology is concerned with the ways a surface can be deformed by twisting, bending, stretching etc. from one shape to another. The topologist is not interested in angles, congruency, distances, etc. but is interested in those properties of a shape that do not change under a given deformation or transformation. Topology is often referred to as rubber sheet geometry. The following diagrams illustrate this.

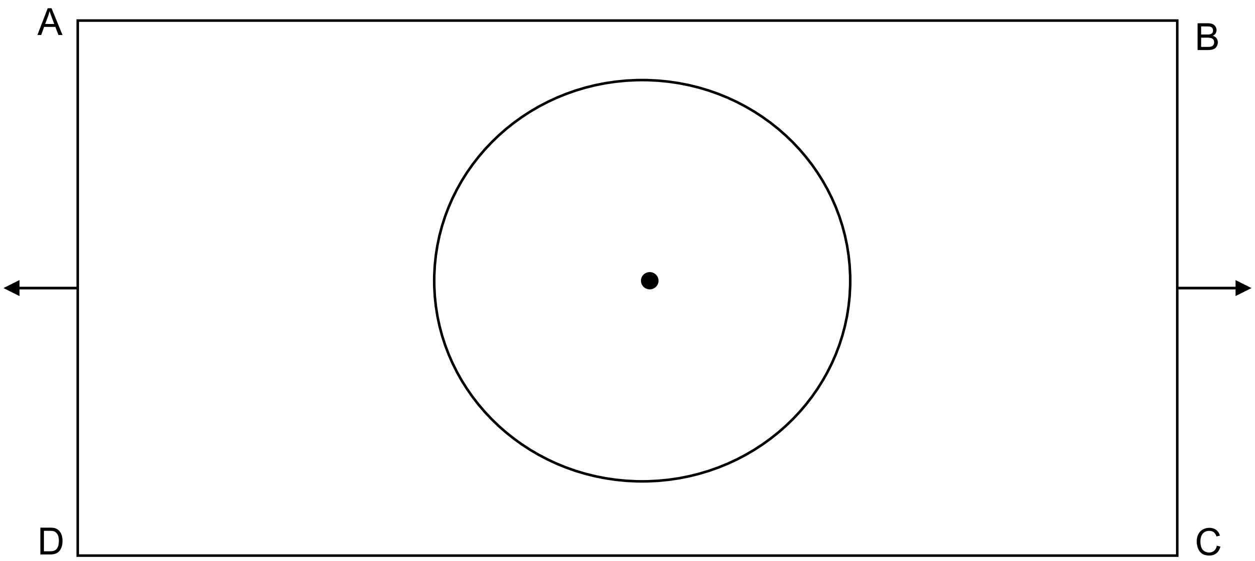

Fig. 33: A Rectangular Rubber Sheet with circle

ABCD (Fig. 33) is a rubber sheet on which a circle is drawn. What will the circle look line if the sheet pulled in the direction of the arrows?

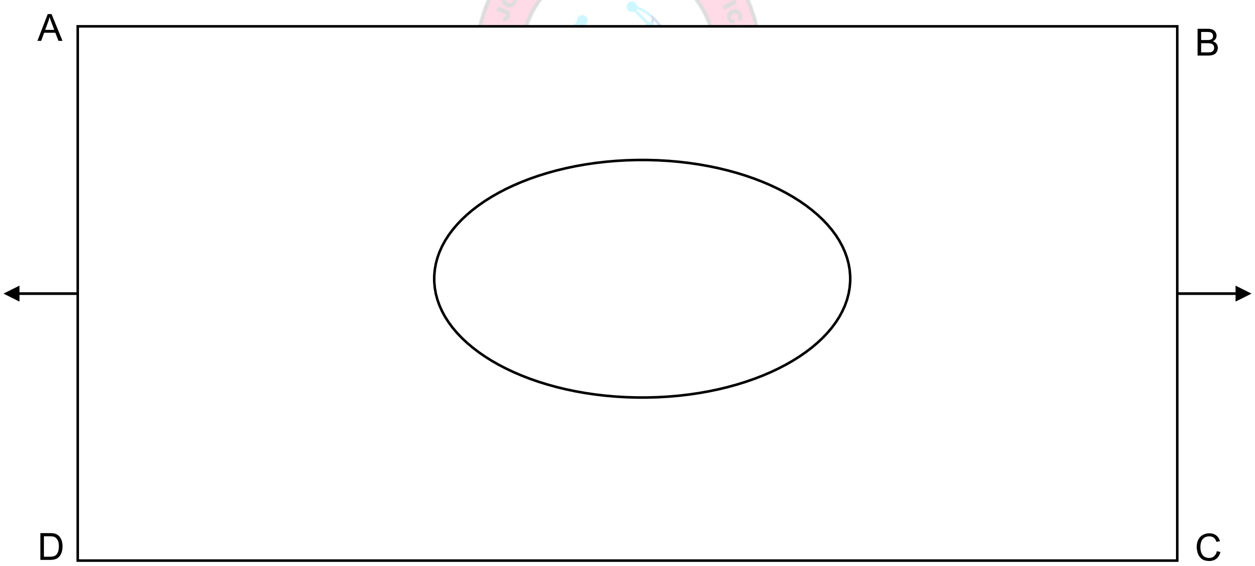

Fig. 34: A Rectangular Rubber with Oval Within

The circle will look like an oval (Fig. 34). To the topologist, this oval is still a circle. It is not had to see that the circle in Fig. 33 can be transformed to any closed shape, viz. square, oblong, triangle, etc. by appropriate stretching of the rubber sheet. Again if one should look at a spherical mirror, one will see an image of one's face much distorted, sometimes beyond recognition. This distortion is a function of the sphere. This function can be stated algebraically. This type of thinking, that is, mathematizing an observable phenomenon has led topology to become a full blown disciplining with diverse applications.

The problem of crossing the seven bridges of Königsberg in one continuous walk without re-crossing the route at some point drew the attention of the great mathematician Euler who showed mathematically that such a walk is not possible. He studied a map of Königsberg and then prepared a graph or network of the bridges and the pathways shown below.

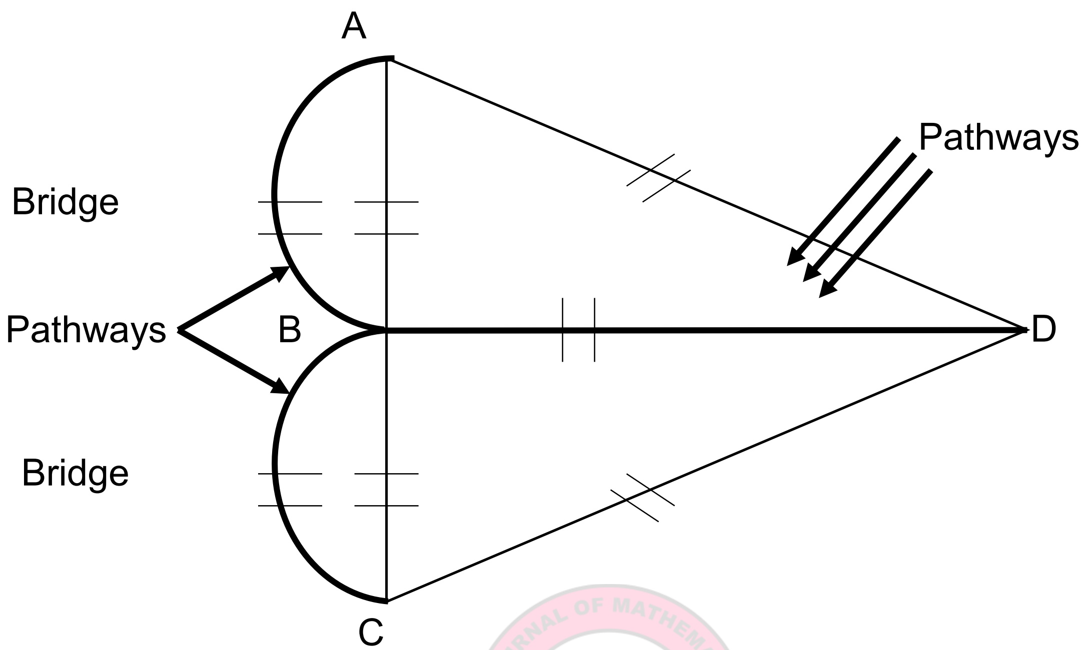

Fig. 35: Graph of the bridges of Königsberg

A, B, C, D are the points where the pathways converge. Euler studied this map carefully and it is suspected that he may have studied other maps which he possibly invented before he came to the general conclusion that it is not possible to cross a network with three or more points at which an odd number of pathways converge without retracing a path. This work of Euler gave birth to a full blown discipline called Graph Theory which has application in electrical circuitry, the laying of electrical cables, water pipes and roads, the allocation of flight paths for aircraft and the sailing routes of ships, assist in interpretation of the phenomena of balance, clustering and transactional analysis in Social Psychology. It is worthwhile mentioning that Graph Theory provides powerful mathematical models (like the other Euler used to study the crossing of the bridges) to study, to interpret and to solve problems pertaining to phenomena of the physical world as well as those of the world of human interactions with humans.

Euler made another important discovery. On examining convex polyhedral (a polyhedron is a many sided solid shape like a cube), he discovered a relationship between the verticles, edges and faces. On examining the cube or cuboid, triangular prism, tetrahedron, square pyramid, octahedron etc. Eule tabulated his findings like this:

Solid	**Faces**	**Vertices**	**Edges**
Triangular Prism	5	6	9
Cube	6	8	12
Cuboid	6	8	12
Tetrahedron	4	4	6
Square Pyramid	5	5	8
Octahedron	8	6	12

Euler searched for a principle that underlie all these solid shapes and found that the sum of the number of Faces and Vertices minus the number of Edges is always 2. This can be stated succinctly thus:

$$F + V - E = 2$$

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It is possible to construct a convex polyhedron if the above relationship between faces, vertices and edges does not hold!

11. STILL MOVING BEYOND EUCLID

Projective Geometry – the Geometry of Artists and Architects

A picture on a canvass is a two dimensional thing, yet we get the impression o the feeling of depth. The artist uses the theory of perspective to achieve this. The following diagram illustrates this. The telephone poles are placed at equal distance on a straight line on either side of the road.

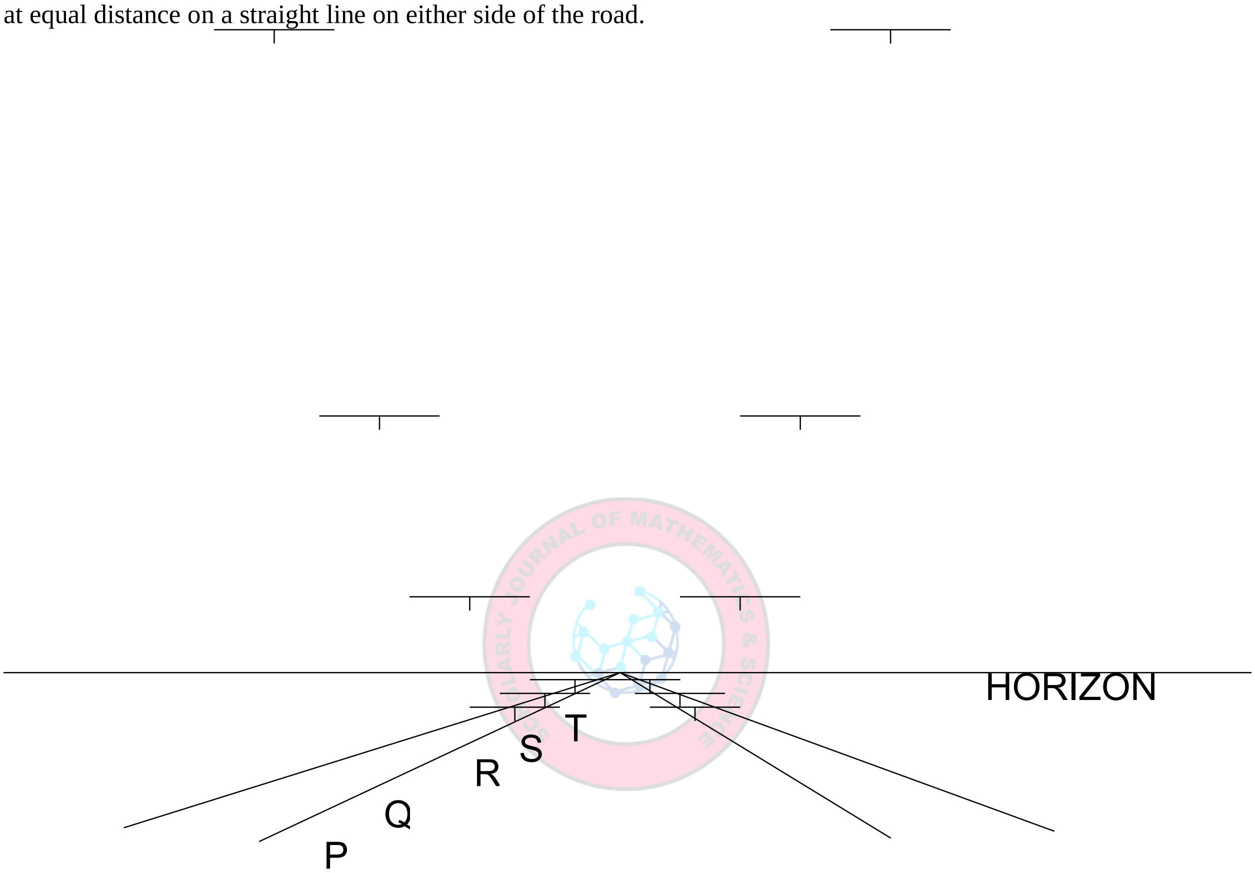
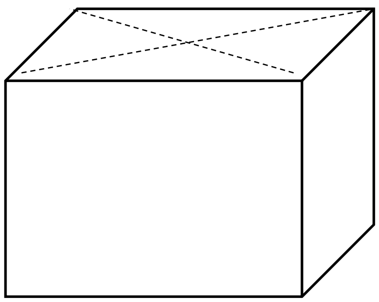


Fig. 36: Lines of perspective

At once we get the feeling of depth. The spaces between the poles appear to get shorter and shorter as they recede in eth distance and finally disappear in a point on the horizon.



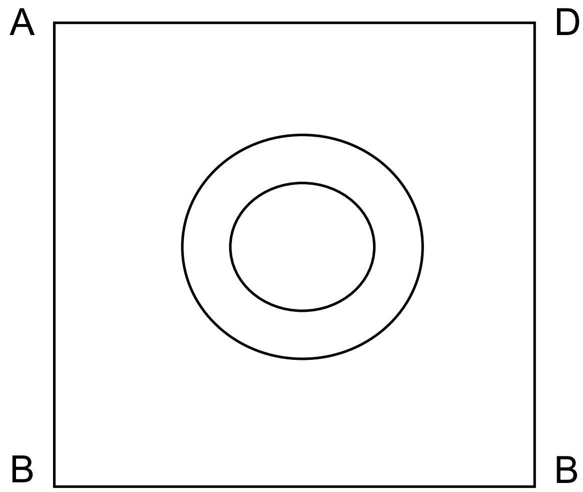


Fig. 37: (a) a Cube at eye level (b) A playing field

Figure 37(a) shows a cube as seen from eye level. The top faces does not appear to be square when in fact it is a square. So angles do not look like right angles and the diagonals do not bisect each other. Figure 37(b) is an aerial picture or both photograph of a playing field taken at right angle to the field. Figure 37 (b) is called an orthographic projection of the playing field but taken at ground level.

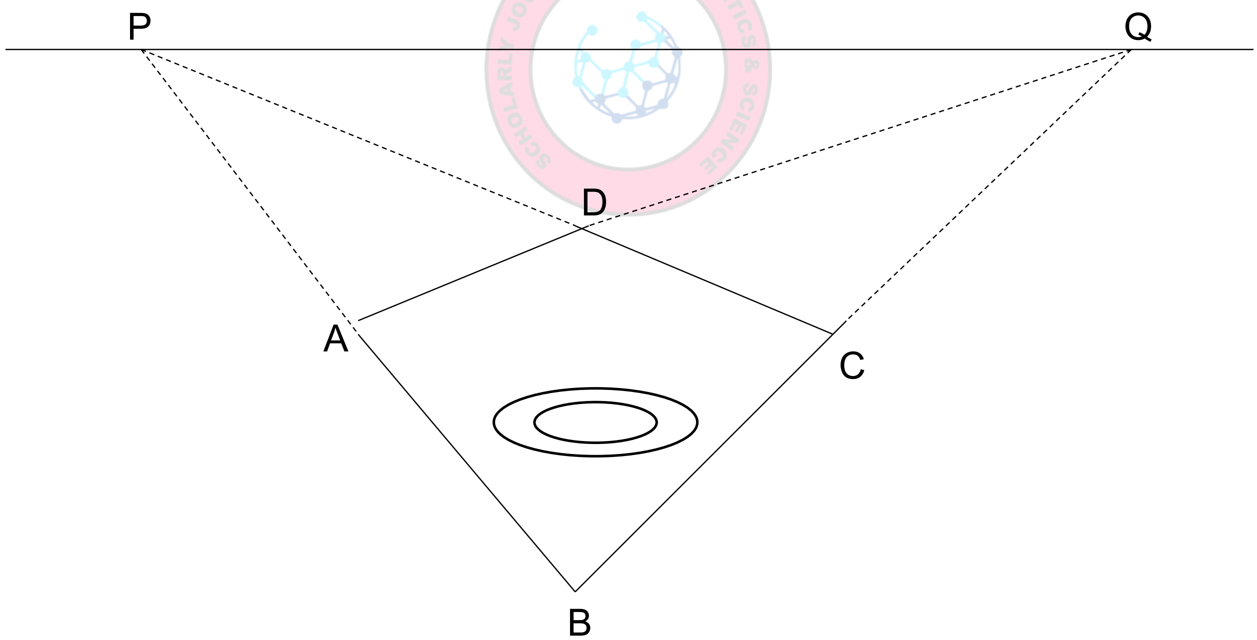


Fig.38:

We observe that in an orthographic projection points, lines and angles are preserved. Once angles are preserved, the mathematician says that object and photo are similar and direct ratios are observed.

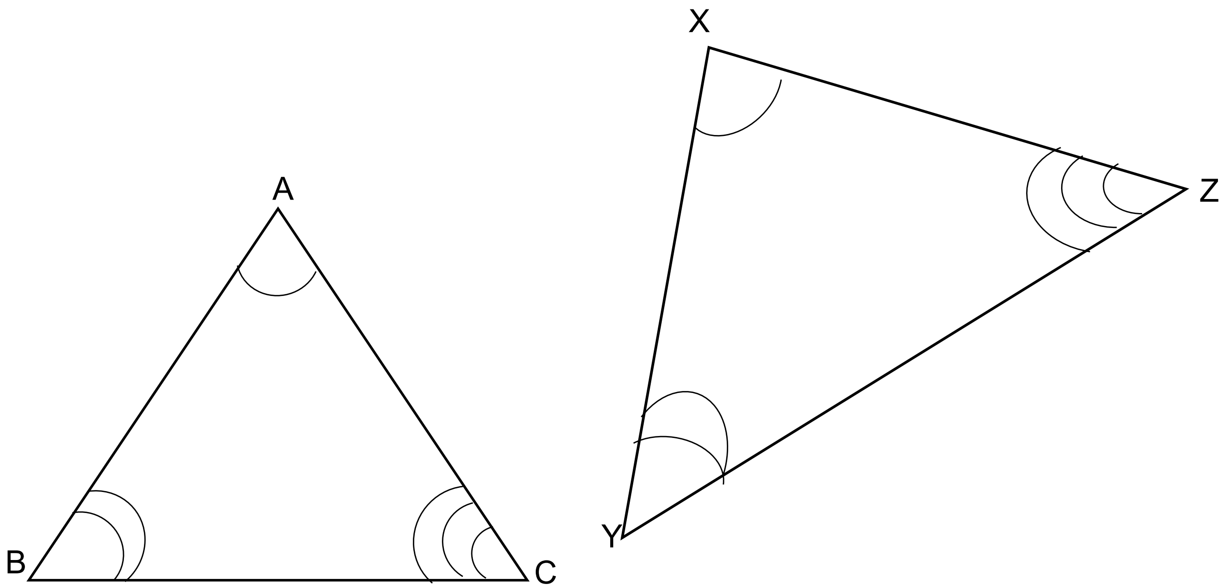


Fig. 39: Similar triangles. Corresponding angles are equal

In the triangles ABC and XYZ, the ratios hold. Considering corresponding lines,

We have $\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}\, or\, \frac{AB}{AC} = \frac{XY}{XZ}\, or\, \frac{AB}{BC} = \frac{XY}{YZ}$ (corresponding ratios).

In a non-orthographic projection lines are preserved but not their lengths, points are preserved, angles are preserved, but not their sizes. The properties of an object in a drawing or photograph that are preserved are called projectives. The study of these properties form the subject matter of Projective Geometry.

Figure 38 does not look like Figure 37; it looks a bit distorted just as it would look if viewed from the ground, but we got the feeling of depth. Length, angles, parallelism, length ratios are altered, but not projective properties.

If AD and BC in Figure 38 are produced they seem to met at Q, a point on the horizon. And if BA and CD are produced, they seem to meet at P on the horizon. P and Q are imaginary points, but to the mathematician they are mathematically real points. Lines that meet at a point are referred to as concurrent lines. This idea helps the mathematician to unify theorems. We see Figure 38 lines AD and BC are parallel and so are the lines BA and CD. The mathematician considers these parallel lines to meet at infinity. This is certainly contradictory to experience, yet it produces true results!.
An interesting result emerges from the following Figure 40.

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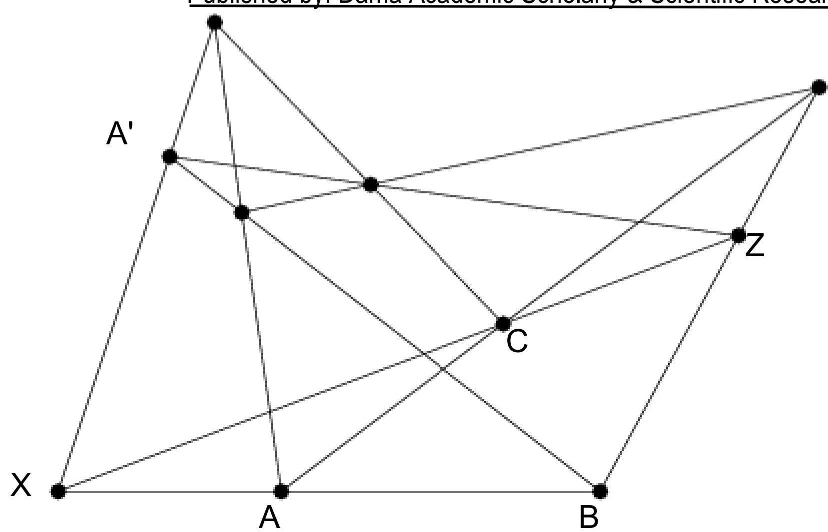
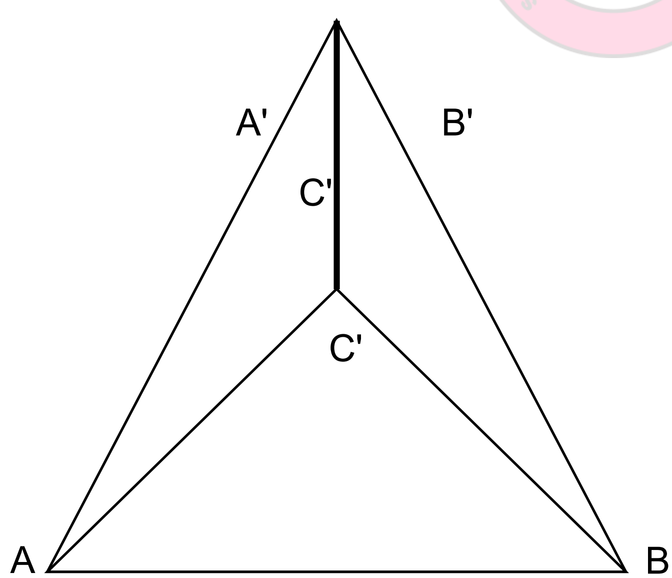


Fig. 40: Desargues configuration in 3 dimensional space

From O draw three lines OA, OB and Oc, where A,B, C are points on a plane (or on the ground taken to be flat). Select points A', B', C' on OA OB and OC respectively. Produce BA, BC and AC. Now produce B' A' to meet BA produced in X, produce B' C' to meet B C and produce Y and produce A' C' to meet AC produced in Z. X, Y, Z lie in a straight line. This result does not depend on the ideas of straight line and points and therefore it is fully projective. This result is known as Desargues' Theorem after G. Desargues (1593-1662) who was an engineer and architect. It is in Desargues' work the germ of projective geometry was sown. Now it is a full blown discipline, clear, sharp and logical unlike Euclid geometry. It is interesting to note that Euclid geometry is the starting point for Projective Geometry, but it in time it rapidly moved away from Euclid.

We refer to Figure 41 (a). OABC is a triangular pyramid or tetrahedron, to be truncated along the plane A B C.



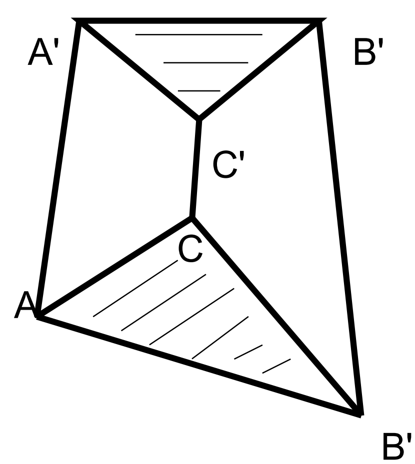


Fig. 41 (a) Tetrahedron **(b) Truncated tetrahedron**

BA and B' A' produced will meet at some point on the plane on which the tetrahedron rests. So will BC and B' C' produced meet, and AC and A' C' produced meet. These three points lie on a straight line. This simplifies the theorem of Desargues. This theorem may be demonstrated by using a solid model of the tetrahedron truncated by the plane A' B' C'.

Line ratios of an orthographic projection or picture of an object apply, but a picture of an object taken at an angle not 90° show lengths and angles altered.

What ratio holds here?

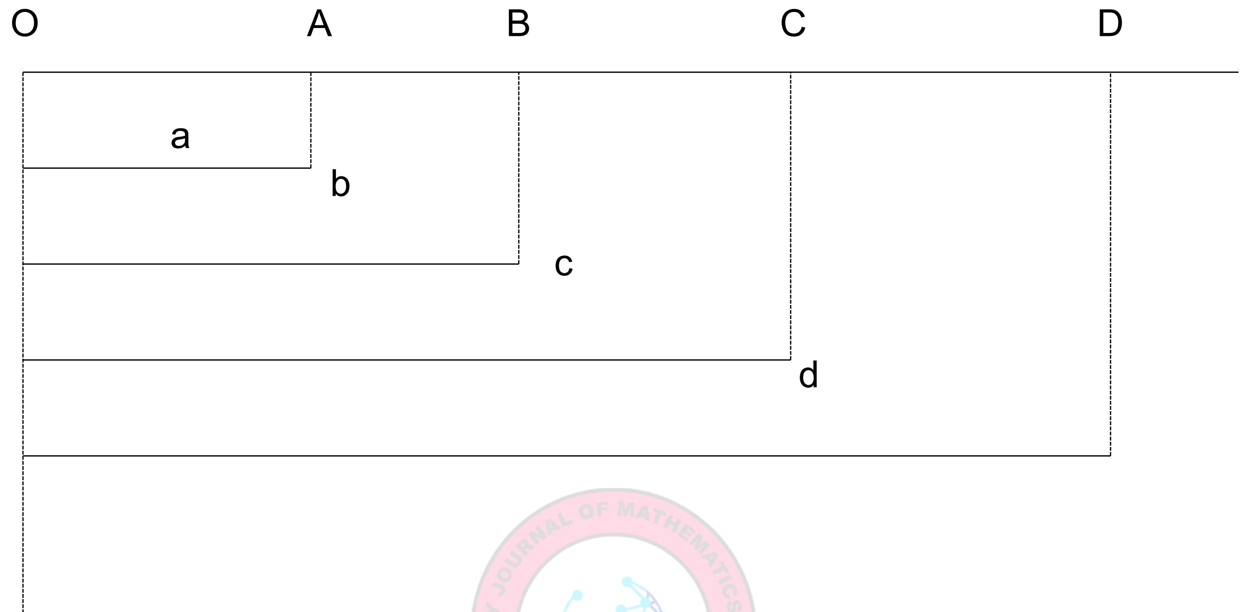


Fig. 42: Cross ratios

Figure 42 shows that A, B, C,D lie on a straight line and that their distances from O, a reference point, are shown as a, b, c, d, respectively. There is a number, say x, that expresses a ratio about the distance of A, B, C, D from O. It is here that

$$x = \frac{(a-b)(c-d)}{(b-c)(d-a)}$$

This is called a cross-ratio and is due to Pappus. The above is not an obvious result and it is not known what train of thought led to this. This quantity, x, is exactly the same for the picture as for the real object. From Figure 36, we measure the distances between the poles P, Q, R, S, T, andcalculate the value for x in the alone expression. We now measure the actual distance between the poles and use this measurement to find the value of x. It will be found to be the same! Cross-ratios of objects and their pictures are invariants, that is, no matter what perceptual changes take place in a picture of an object, this cross-ratio always holds true.

Perspective Geometry emerged from art, photography and architecture and has practical applications in these fields. But its influence does not stop here. Different kinds of differential equations require different methods for their solution. A theory developed by Sophus Lie (1842-1899) establishes a single principle that underlies all the different type of differentiated equations. This theory was developed from geometrical questions closely related to Projective Geometry. This is only one example of many in which Projective Geometry pays an important part.

12. RANDOM CHAOS. IS THERE A CHANCE?

If a coin is tossed a head (H) or a tail (T) may show up. The mathematician says that the events H and T are equally likely and that {H, T} is the probability space. The probability space is the set of all possible events, and in this case it is H and T – two possible events. In a single toss of a coin it is not possible to predict the outcome. But there are two possibilities, H and T, but not both. The mathematician reasons that one event, either H or T will occur out of a possibility of two events, and he states that the possibility of either H or T not both showing up is one in two in a single toss of a coin. Therefore the probability of H occurring is ½ and T occurring also ½.

What is the probability of a 3 occurring in a single throw of a die? We go with the mathematician. List all the possible events. This is the probability space , i.e. {1, 2, 3, 4, 5, 6}. There are 6 possible outcomes, each equally likely to occur. There is one event in six possible events, so we write the probability of a 3 showing up in a single throw of a die is $\frac{1}{6}$ This is so for all other numbers in the probability space.

Suppose a coin is tossed 100 times, what are the outcomes? If HHT HHT HHT… occur then it is easy to predict the next event. This is so because the events occur in a pattern. Patterns define order and order is the essential principle of cosmos. In practice, however, this may occur – HTT HT HHH TT HTH TTT TH … There is no pattern in these outcomes or events. It is possible to predict what the next outcome. The above outcome is definitely random. This randomness continues for an infinite number of tosses. No prediction about the occurrence of an event or the non-occurrence of an event can be made. Randomness defines chaos.

Since in a random occurrence of events, no prediction can be made, the mathematician compares the occurrence of H to that of T by a ratio. If there are n tosses of a coin and H(n) refers to the occurrence of H and T(n) to that of T, then the ration $\frac{H(n)}{T(n)}$ compares the occurrence of H to that of T. This is one of the many structures mathematicians use to study randomness.

It has been found empirically what when n is finite the ratio H(n)/T(n) . That is $\frac{H(n)}{T(n)}$ is approximately 1. As n gets larger and larger the ratio approaches 1, and making prediction about an event becomes possible. When n is infinitely large the ratio is 1, and H and T are equally likely to show up. In other words the occurrence of the number of H is the same as that of T. Hence the probability of correctly predicting an event is ½ when a coin is tossed an infinite number of times. Out of chaos comes cosmos. Hats off to the mathematician! But does he say what chaos or randomness is, what infinity is? To make sense of randomness he uses the difficult and undefined concept 'infinity' to interpret it. This is untenable, yet this type of thinking of the mathematician produces real, true results that can be successfully used in many areas of human endeavour. However, even at infinity the outcome of the next toss cannot be accurately predicted. The mathematician says that the chance that H shows up is one in two. But even this might not happen.
What are the outcomes if two coins are tossed once? Let us see.

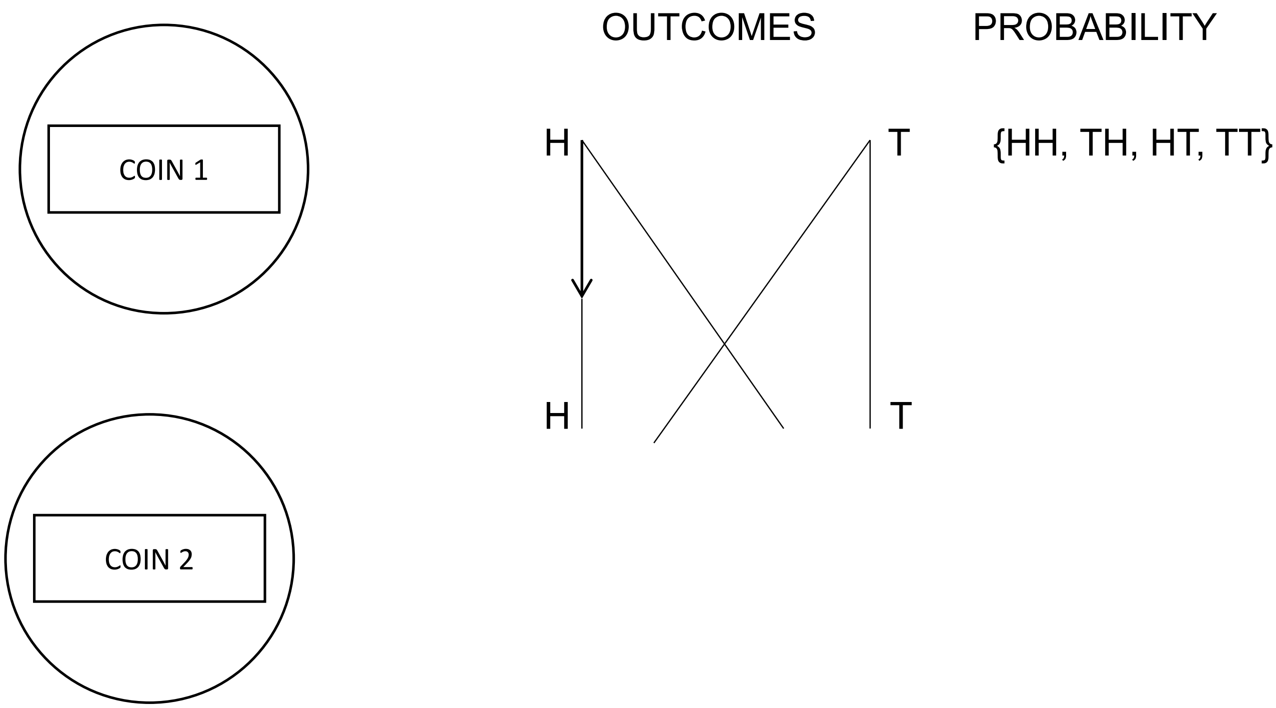


Fig. 43: Tossing of two coins

The outcomes are that HH may show up or HT may show up, or TH may show up or TT may show up. Then what is the probability that 2 heads may show up? There are four outcomes and the occurrence of 2 Heads is one among four outcomes or events, i.e. ¼ . Another way to think about this is as follows. If one coin is tossed the probability of a head showing up is ½ . If another coin is tossed the probability of a head showing up is ½ . If both coins are tossed the probability of two heads showing up is ½ x ½ = ¼, the same as above.

When more than two coins are tossed, finding the probability space is tiring and may become quite messy when the method described in Figure 43 is used. It does not take much imagination to see what may happen when 10 coins are tossed. This is a challenge to the mathematician. He studies Figure 43 and observes the outcomes. The event of two heads showing up is a combination of HH, the event of a head and a tail showing up is a combination of HT or TH, for tail and head, and that if two tails are a combination of TT. What known model will fit this? The Binomial expansion. Let us see this model in action.
$(H + T)^2 = H^2 + 2HT + T^2 = HH + 2HT + TT$ or $HH + HT + HT + TT$ (4 events), where the index 2 stands for the number of coins.

When two coins are tossed, what is the probability that two tails will show up?

It is ½ x ½ = ¼ i.e. one in four.
What is the probability that a head and a tail will show up? It is 2 x ½ x ½ = ½ , i.e. one in two. This is obvious from Figure 43. Let us see what will happen when three coins are tossed.

$(H+T)^3 = {}^3C_0H^3T^0 + {}^3C_1H^2T^1 + {}^3C_2H^1T^2 + {}^3C_3H^oT^3$, where ${}^3C_2$ is a combination of 3 things (coins in this case) taken two at a time and so on.

Thus:

$$(H+T)^3 = \frac{3}{0!}H^3 + H^2T\,(\frac{3}{1!}H^2T + \frac{3.2}{2!}HT^2 + \frac{3.2.1}{3!}T^3 = H^3 + 3H^2T^1 + 3H^1T^2 + T^3$$

What is the probability that 3 tails will show up if 3 coins are tossed? The expansion shows T3 or TTT. The probability of T showing up when one coin is tossed is ½. Therefore when three coins are tossed it is ½ x ½ x ½ = $\frac{1}{8}$, i.e. the probability of TTT showing up when 3 coins are tossed is one in eight. What is the probability that two heads and a tail will show up? It is $3H^2T$ from the expansion and this is 3 x ½ x ½ x ½ = $\frac{3}{8}$. What is the probability that one head and two tails will show up? Well try it. Do as the mathematician does. What is the probability that six expectant mothers may have four girls and two boys? A mother can have either a boy or a girl, but in a rare case she may have both, or she may not have any, but we assume that each mother will have either a boy or a girl. We use the Binomial expansion because we have only two elements viz., boy and girl. Thus (G+B) is a binomial expression. So
$(G+B)^6 + {}^6C_0G^6B^0 + {}^6C_1G^5B + {}^6C_2G^4B^2 + {}^6C_3G^3B^3 + {}^6C_4G^2B^4 + {}^6C_5GB^5 + {}^6C_6G^0B^6$

Where the index 6 refers to the 6 expectant mothers.

The probability of the occurrence of 4 girls and 2 boys is given by ${}^6C_2G^4B^2$. Now the probability of one woman having a girl is ½ and boy is ½. Thus the occurrence of 4 girls and 2 boys from six expectant mother is $\frac{6.5}{2.1}x\frac{1}{2}x\frac{1}{2}x\frac{1}{2}x\frac{1}{2}x\frac{1}{2}x\frac{1}{2}=\frac{15}{64}$.

What is the probability that the six mothers may have 3 boys and 3 girls? Surely you will use ${}^6C_3G^3B^3$ in the above expansion. Magnum est Mathematics.

Here is another problem and let us see how the mathematician approaches a solution.

The local cricket team has to play a match at home on a particular Saturday in April. In dry condition the team wins 4 out of 5 matches, but in wet conditions the team has a 50-50 chance of winning at home. The weather records show that in April an average of 10 out of 30 days are wet. On Sunday the sports news on T.V. announced that the team won the match. What is the probability that it was wet on Saturday?

The mathematician analyses the problem by studying the information given in the problem. All irrelevant information is not considered. He interprets what is given as follows.

The probability that the team will win on a dry day is $\frac{4}{5}$ (4 out of 5).
The probability that the team will win on a wet day is ½ (50-50 chance).
The probability of a wet day in April is $\frac{1}{3}$ (10 out of 30).

If they won what is the probability that the day was wet? There are three probability situations here and an intuitive approach may not help. A common sense approach used by many beginners runs like this. The probability of a team winning on a wet day is ½, and the probability of a wet day occurring is $\frac{1}{3}$. Therefore the probability of the team winning on a wet day is $\frac{1}{2}x\frac{1}{3}=\frac{1}{6}$. It sounds logical but it is not true because the total possible outcome was not considered. The probability space is }{Dry : Win, Wet: Win} and we need to find the probability of (Wet/Win) i.e. the probability that the day was wet given the team won. The probability of a dry day is $\frac{2}{3}$.

The probability of winning on a dry day is $\frac{4}{5}$.

Therefore, the probability of winning on a dry day is $\frac{2}{3} x \frac{4}{5} = \frac{8}{15}$.

The probability of a wed day is $\frac{1}{3}$.

The probability of winning on a wet day is $\frac{1}{2}$.

Therefore the probability of winning on a wet day is $\frac{1}{3} x \frac{1}{2} = \frac{1}{16}$.

If we write P($\frac{Wet}{Win}$) to mean the probability that the day was wet given that the team won then

$$P(\frac{Wet}{Win}) = \frac{P(Wet:Win)}{P(Dry:Win) + P(Wet:Win)} = \frac{\frac{1}{6}}{\frac{8}{15} + \frac{1}{6}} = \frac{\frac{1}{6}}{\frac{21}{30}} = \frac{5}{21}$$

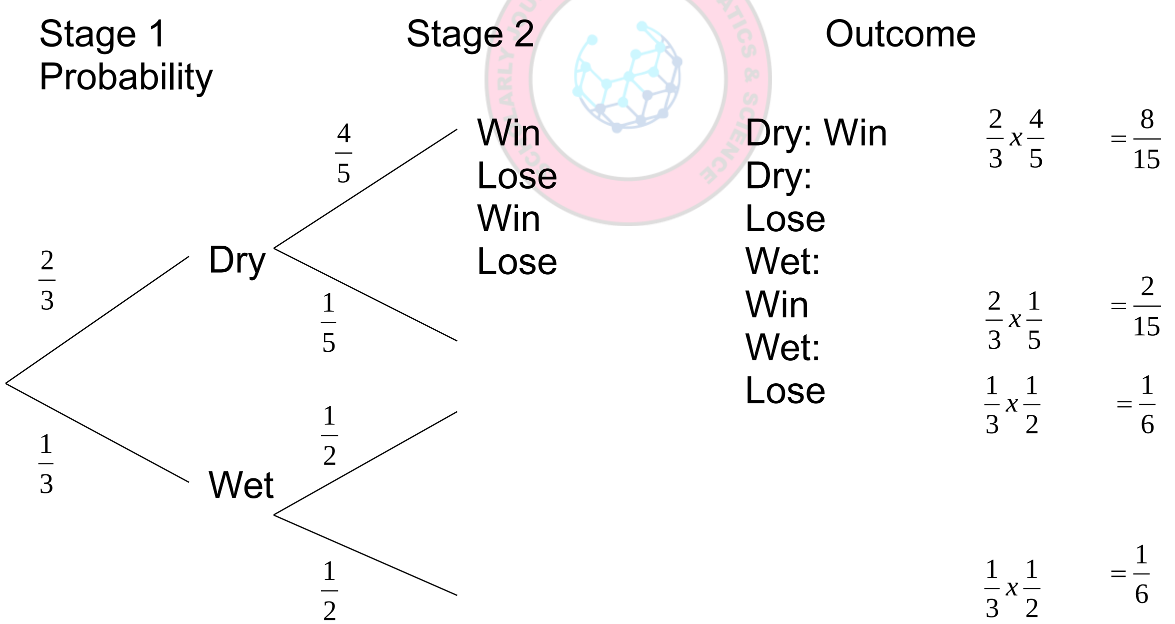


Fig. 44:

The above Figure 44 is simple and quite versatile. Many questions pertaining to the team and weather conditions can be answered using it. For instance, what is the probability that the day was dry given that the team lost? Take your chance; it is rewarding!

13. MOVING ON

Mathematics is a work of art in logic. It grows out from the depths of situations that are ambiguous, contradictory and paradoxal. Through insight, through intuition, through logic, the mathematician discovers mathematics and invents a most perfect symbolism to express it. It is this symbolism the mathematician uses as a veritable tool to investigate ideas, ambiguities, contradicting and paradoxes, and in so doing discover a body of mathematical knowledge. This knowledge further scrutinized reveals laws that define the nature of mathematics. These laws are applied to new situations even if these situations are ambiguous, contradictory or paradoxal and so new mathematics is discovered, and on examining this new mathematics, using it, the mathematician discovers new ways to use this mathematics and so new knowledge of mathematics is advanced. This process goes on and on. This is put nicely by Fox (1950) when he wrote: "When relations have been discerned, these may become the fundamentals for further relations, and so indefinitely. This is the foundation of much mathematical reasoning, for the relations between certain symbols are themselves symbolized and thereafter become foundations for further relations".

The question whether mathematics was discovered or invented occupied the minds of philosophers and mathematicians for centuries. The question is still pertinent today. The Platonic school of thought contends that there is a body or world of mathematics where existence is different from physical existence and mental existence. Yet the world of mathematics is mysteriously connected to these two forms of existence. Mathematicians and scientists have tapped only a very small portion of this mathematical world to interpret and explain the physical world. This world of mathematics is considered to be absolute and complete. And nothing the mathematician discovers is outside this world. The works of pure mathematicians have extended the realm of mathematics to such an extent that there is no obvious connections with physics or any other science with this extended mathematics, though occasionally unexpected applications crop up. The Platonic school contends, then that mathematicians only discover mathematics that is already there; they do not create it! Is there mathematics in this Platonic world of mathematics that is beyond human reasoning and by extension beyond discovery?

We may ask the question: Was language discovered or invented? Humans are endowed with the ability to reason, and with a complex vocal chord that is capable of an infinite variety of sounds. These two endowments distinguish humans from animals. From the urgent need to express thoughts that include names, ideas, beliefs, feelings, intentions etc. sounds were developed or invented and collectively agreed upon to represent them. We may well imagine that at first these sounds were monosyllabic utterances, but rapidly developed into complex forms in which these sounds or words were chained into sentences to make more complete sense than mere grunts and simple sounds. As time rolled by and human thoughts become more and more complex, more sounds or words were invented to express them and so a formal language was developed. The need to record and communicate thoughts non verbally led to the invention of the alphabet and writing, perhaps humans' greatest invention. There is experience , there is a thought or feeling or an intention. Language tells us how to say or write them down.

Mathematics is a language, and like all languages, it sprung up from the imperatives of communication. Twoness was always with us. Two hands, two eyes, two ears. Oneness was there. One nose, one head, one mother. Fiveness was there. Five digits on one had and five on the other. And repetition of manyness was there. Five petals on certain flowers, and this did not vary for those kinds of flowers. Nature is replete with manyness. From many universes to many electrons. To express manyness a verbal language was invented, and then symbols were invented to express and record manyness or numbers. Counting began and recording of this activity become a daily affair. Men of numbers soon discovered that certain counting numbers are multiples of 2 and others are not. Another class of number was discovered. These numbers are so formed a class or set. By classifying numbers according to some discovered properties, men of numbers or as we call them mathematicians were able to study numbers more reflectively and so discovered more secrets of numbers. The need to find the total number of members of two or more sets mathematician had to invent an operation to do this. This addition, and so forth for subtraction, multiplication and division. Applying these operations on numbers revealed further properties not only of the numbers themselves but also of the operations. These properties of numbers and operations form the basis of all mathematical knowledge. From the above discussion it appears that mathematicians invented the language and processes that they use to pry open the treasure vault of mathematics.

History tells us that there are several instances where mathematician had to invent mathematics to solve problems. To find the area of a circle the Greek used a device (already discussed in this paper) where they considered the area of a circle the sum of very small sectors whose areas are almost zero. And this produced real and true results! A small sector can be considered a small triangle. The method to find the area of a triangle was discovered. So the invented device to find the area of a circle is based on the discovery of the method of finding the area of a triangle. In order to study rates of changes as observed in nature, Newton and Leibniz invented a mathematical device called calculus. This is based on the concept of infinitely small increases in rates and the undefined concept of infinity. Were these concepts discovered or invented? Claude Shannon used Boolean algebra to work on his theory of information and communication and it is said by many experts that he created the mathematics he needed besides the Boolean algebra. (Darling, 2005).

The preceding paragraph gives us a brief glimpse of the double perspective of discovery and invention of mathematics. This is indeed an ambiguous situation. Careful examination of the development of mathematics reveals that discovery and invention are not disjoint; they do have areas of commonality, and that they complement each other. "Together they describe mathematics as a vast dynamic field of creative activity that is imtimately related to 'truth' that is ever changing and ungraspable in any fixed and absolute network. Mathematics is always reinventing itself." (Byers, 2007). Or is mathematics rediscovering itself? One school of thought contends that mathematicians invent tools to probe deep into the inner secrets of mathematics and in so doing discover 'truths'. Invention and discovery are both creative activities, and these activities are moving on hand in hand. Yonder horizon this is new mathematics.

14.0 NOTHING THAT IS SOMETHING

Before the sixth century A.D., zero did not make its appearance in the continuing arena. Numerals were already invented to represent one, two, three, etc., but represent nothing, no symbol was yet invented. The concept of empty or nothing existed as this was an inevitable part of the human experience. The number of beads in the columns of an abacus could be represented by numerals, but there was no numeral to represent the number of beads in a column that had no beads. It became imperative that a symbol be invented. The earliest record of the use of zero was found in India around 458 A.D. and as early as the year 594 the Indians started to use a decimal place- value system. For instance, in the number 13, 1 stands for 1 ten and 3 for three ones. So the 1 in the number 13 represents a value greater than that represented by 3. But there was no numeral to represent the number in a place that had no entry! The Indians used a dot or Bindu to represent no entry or nothing. Thus to write two hundred and five, they wrote 2.5. It is interesting to note that the Bindu or point represents a mystic concept.

Bindu is the nothing from which everything emerges. Bindu is a potential state just waiting for things to happen. It is in this state of nothingness that everything resides? We are told that God created the universe and all that are there in them from nothing. Is Bindu that nothing? The Existentialist philosophers look at this concept of nothing from the other end of the telescope. They contend that everything eventually is reduced to nothing. This reminds us of a law in thermodynamics which states that everything runs down, reach a state of entropy or disorder and then becomes nothing. The Indians eventually replace the Bindu by Sunnya which is symbolized by 0. Of course we call it zero. Sunnya means empty or nothing. Thus zero sprouted out from an urgent need as a place holder in a place-value system of numeration. Bertrand Russel contends that zero was forced upon mankind, but is this really so?

Zero represents the nothing that is. What is nothing? Can we define 'nothing'? It is hard to define 'nothing' as " :'Nothing' is … . But it is easy to define a triangle as : A triangle is a closed figure with three sides. The word 'nothing' seems to be an absolute term and to define it as we define triangle is not an easy task. The definition of triangle includes the class or genus to which it belongs, i.e. the class of closed figures, and also the property that distinguishes or differentiates it from other closed figures, that is, it has three sides. Genus et differentiam is the maxim for logical definition. Nothing is the absence of something. A presence that implies an absence. Nothing is only when something was. Nothing is referenced by the absence of something. Can we say that something exists only when nothing does not exist? Or do they both exist simultaneously? Or is it true that one implies the other?

The above reflection shows that zero has double meanings: it is and yet it is not. But this is contradictory. The word 'zero' or symbol 0 is a presence that stands for numerical nothing. But is zero itself nothing? In other words is the concept of 'nothing' nothing? Two refers to tangible things e.g. two eyes, two trees, two mangoes etc. To what tangible things does zero refer? Zero is indeed a complex, ambiguously contradictory concept. Yet without this concept Mathematics would have stagnated.

The above question appears to be gibberish, nonsensical or just a play on words, but they carry deep philosophical connotations which are not addresses in this paper. By the year 628 the astronomer/mathematician Brahmagupta spelled out the Arithmetic rules for zero when he wrote:

When sunnya is added or subtracted from a number, the number remains unchanged; and a number multiplied by sunnya becomes sunnya. Thus zero has been redefined in terms of mathematical operations additon, subtraction and multiplication. This definition of zero gives it the status of a numeral like 1, 2, 3 … etc. but it behaves differently from them. As Brahmagupta showed that when two or more numerals are added the result is another numeral different from those that were added e.g. $6 + 3 = 9$. But when a numeral is added to zero is subtracted from the number $6 - 0 = 86$. Multiplication of a number by zero reduces the number to zero. This is a unique property of zero. In summary we write in general terms that if a is a number, then

However, the division of a number by zero posed a challenge. First it was thought that it was not possible. Take for instance 10 ÷ 0. We ask how many set of two are there in 10. We get an answer. Now 10 ÷ 0. We ask how many sets of zero are there in 10? No answer is forthcoming. Thus division by zero is taboo. But we can pose another question. How many sets with no members can be formed from a set of 10 things. Here the answer is 1. But this is not a question of division; it is a question of making sets from a given parent set. From a set of 10 elements, we generate or form 2 sets, one of which is set with no element, that is, in mathematical language the acrdinality of this set is 0. Here zero is not only a place holder in a place-value number system, but a main supporting concept in the development of set algebra and other algebras. It is easy to show that all subsets formed from any parent set includes the empty set – cardinality 0. It appears that zero is omnipresent in sets, stated or implied.

It is possible that Brahmagupta might have wrestled with the idea of dividing a number by zero. He did not dismiss this idea as impossible. He look at, say $\frac{a}{o}$, from an angle. Let us take $\frac{1}{x}$ as an example.

If $x = \frac{1}{2}$ then $\frac{1}{x}$ becomes $1\frac{1}{2}$ = 2 halves

If $x = \frac{1}{10}$, then $\frac{1}{x}$ becomes $1\frac{1}{10}$ = 10 (ten tenths)

If $x = \frac{1}{1000}$, then $\frac{1}{x}$ becomes $1\frac{1}{1000}$ = 1000 (thousandths) and so on.

But a x becomes smaller and smaller the number of unit fractions become larger and larger. This has already been discussed in this paper. As x approaches zero, i.e. becomes infinitely small the number of unit fractions also become infinitely large, though each unit fraction is infinitely small. Brahmagupta used the idea to define infinity as a/o. A bold and powerful step in the development of mathematics.

Philosophers says that infinity is without end, eternal, immortal, self-sustaining, self-renewable, the apeiron of the Greeks, 'that which is, but is beyond thought and therefore beyond words.' (Swatashvatar Upanishad). This definition is of little use to the mathematician except that 'infinity is without end'. Infinity has boggled the minds of mathematicians for centuries and perhaps it still does in spite of the great world of Cantoron this idea of infinity. Can infinity be defined as infinity is …a member of a class of things with a property that differentiates it from the other members of the class? This formal definition is difficult to come by. Sawyer quotes a school boy as saying that infinity is where things meet, but don't. Quite interesting, an innocent statement, but philosophically deep. This shows the contradictory nature of infinity, yet it gives life to projective geometry and art.

Back to $\frac{a}{o}$ as infinity. We repeat the treatment of $\frac{1}{x}$ with a slight change.

If x = 2, then $\frac{1}{x}$ becomes ½

If x = 10, then $\frac{1}{x}$ becomes $\frac{1}{10}$

If x = $\frac{1}{1000}$, then $\frac{1}{x}$ becomes $\frac{1}{1000}$ and so forth.

As x approaches say 10^{100} , we have the situation $\frac{1}{10}$ ^{100} . This represents an infinitely small number – so small that it can be considered 0 in size. But 0 represents nothing, yet in this case it is made to represent something. Is this an illicit process. Zero is nothing, yet it is something. These two meanings of 1 are contradictory. Yet using each meaning of zero in separate situations produces useful results. So we have $\frac{1}{0}$ = an infinitely large number of infinitely small factions. In short, we write $\frac{1}{0} = \infty$. We will submit to the laws of arithmetic? Let us see $\frac{6}{3}$ = 2, so 6 = 3 x 2, then as $\frac{1}{0} = \infty$, so 1=0 x ∞. Is this tenable? Zero is made to mean something very, very small and infinity is moving on getting larger or smaller without end. Can the product of 0 and infinitely be 1.

Again if x becomes infinitely large, then $\frac{1}{x}$ becomes infinitely small or zero. We write this as 1/∞=0. Once more we see that 1=∞ x 0. The problem here is in the notion of equality. Equality means same as. So 1 is the same as (or has the same value) as 0 and ∞. Zero represents nothing; here it represents something. And ∞ represents moving on without end. One is finite, and 0 x ∞= 1!. This is indeed a curious situation. It may be necessary to replace the equality symbol by another symbol to represent the above equation (which may no longer be equations) involving 0 and ∞. The best we can say at present is that as x approaches 0, $\frac{1}{x}$ approaches ∞, and as x approaches infinity, $\frac{1}{x}$ approaches 0, ∞ and 0 being the limiting values of $\frac{1}{x}$ in each case respectively.

We see that 0 and ∞ are intricably intertwined; they are two sides of the same coin. These two concepts are the foundation stones upon which Modern mathematics is built.

The efforts of mathematicians lies in the quest for truth, to know the truth, and to communicate it. Mathematics is about truth and to glean this out mathematicians developed a language and methodologies to do this. Mathematical truths are established by rigorous proofs that are based on well established procedures. This has been discussed in this paper earlier. But does mathematical truth reside in the proof itself or does it exist independently or the proof? There are instances where several independent proofs establish the truth of a proposition, but they are not the same as the truth of the proposition. They may use the pre-established truths of related propositions to establish the truth of a given proposition. Two important questions we must ask: Are proofs fool-proof? Are there proofs that are flawed?

Some philosophers and mathematicians claim that truth is relative. It holds good within a given field of experience. Man's experience of a flat universe tells him that if a line cuts two or more lines at right angles, then these two or more lines are parallel to one another. This is always true in a flat universe. There is no need for proof; the truth is self-evident as experience dictates. If the experience shifts or broadens, then the truth that holds good for a flat universe shows that the above condition for parallelism does not hold. Mathematical truth is thus relative to a given experience. Logic is not concerned with truths; it is concerned with correct reasoning. Mathematics is concerned with truths that are established through correct reasoning. Truth underlines man's craving for certainty and it is 'more likely to be found in mathematics than elsewhere'. (B. Russel). But then what is truth?

Mathematics has developed from situations that are ambiguous, contradictory and paradoxal. How then can mathematics be true if it is developed from these uncertain situations? But experience shows that there is great certainty in mathematical truth with a given frame of reference; the truth may change if the frame of reference changes as we have just see in the case of the flat universe and the spherical one.

Mathematics has mushroomed into a multidimensional universe of pure thought. It is utilized in almost all field of human endeavour and has become an indispensable tool in the affairs of humans. Maxwell used mathematics to predict the existence of radio waves long before they were discovered, Einstein in an effort to interpret some equations posited that if monochromatic light, the rays of such amplified light will carry a tremendous amount of energy. This theory was formulated in 1905, but it was in the middle of the 1940's that technology was developed enough to demonstrate this. So the laser was born. Laser has diverse uses in engineering, communications, both auditory and visual (CD, DVD, digital technology), science and medicine. The existence of the planet Neptune was posited by mathematics and informed by it where it should be found in the heavens. Recently Fotini Markopoulou Kalamara, a researcher, working on the theory of Loop Quantum Gravity found that abstract loops are composed of matter and space and that some real physical properties of these loops have been derived by pure mathematics which quantifies what space is and at the same time unifies the quantum and gravitational realms. However, at present, there is very little empirical proof of this theory. (Discover: March 2008). Such is the power and beauty of Mathematics.

Mathematics possesses an inner beauty and harmony. From the elegance of patterns, to the beauty, yet awesome rigour of proofs, from the cosmic dance of time to eternity, to the harmony of music, to the infinity where things become nothing – mathematics stands supreme. This is eloquently put by an eminent authority quoted by Fox (1950).

"Mathematics rightly viewed, possess not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure and capable of stern perfection such as only the greatest art can show." Is there a branch of knowledge through which all mathematics can be knows?

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