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CONCAVE AND CONVEX FUNCTIONS

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DECLARATION

WE DECLARE THAT WE HAVE, UNDER SUPERVISION, UNDERTAKEN THE STUDY HEREIN SUBMITTED.

………………………….

DATE

PETER KWASI SARPONG ….………………………..

OWUSU-HEMENG ANDREW …………………………...

I DECLARE THAT I HAVE SUPERVISED THE WORK HEREIN SUBMITTED AND CONFIRM THAT THE STUDENTS HAVE MY PERMISSION TO PRESENT IT FOR ASSESSMENT.

…………………..

……………………

DATE

MR. ACKORA PRAH

DEDICATION

To our parents, siblings and all our loved ones. And to all our lecturers at the Department of Mathematics, KNUST for their support, dedication and patience.

ACKNOWLEDGEMENT

As an act of showing our appreciation, we would like to recognize the following people for all their unconditional and selfless support in bringing up this project. First of all, to Him who crowns all with glory, for His love, guidance and mercy and making them manifest in our lives. Next is to our supervisor, Dr, Joseph Ackora - Prah who made sure things were rightly done, we say thank you. Also to our parents for their financial support, we are very grateful to them all. And to all who helped us in one way or the other, we say thank you.

ABSTRACT

In recent years, convex optimization has become a computational tool of central importance in mathematics and economics, thanks to its ability to solve very large, practical mathematical problems reliably and efficiently. The goal of this project is to give an overview of the basic concepts of convex sets, functions and convex optimization problems, so that the reader can more readily recognize and formulate basic problems using modern convex optimization. This helps in solving real world problems.

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CHAPTER ONE

1.0 INTRODUCTION

Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non linear programming problems. It is very important in the study of optimization. The solutions to these problems lie on their vertices.

The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test. Miraculously, this has a natural generalization for the several variables case, the Hessian test.

This project is intended to study the basic properties, some definitions, proofs of theorems and some examples of convex functions. Some definitions like convex and concave sets, affine sets, conical sets, concave functions shall be known. It will also prove that the negation of a convex function will generate a concave function and a concave set is a convex set. There is also the preposition that the intersection of convex sets is a convex set but the union of convex sets is not necessarily a convex set.

This work is intended to help students acquire more knowledge on convex and concave functions of single variables. This will be done by differentiating the given function twice. If the second differential of the function is positive then we have a convex function. On the other hand if the second differential is negative then that function will be considered as concave. Examples will be solved to elaborate more on this. The convexity of functions of several variables will also be determined. This will be done by the use of the Hessian matrix. This will generate the idea of principal minors and leading principal minors. Firms can also use the idea of convex functions to know how they are doing in the market. Equations can be generated and with the help of curve sketching they will know if they are maximizing profits or making losses.

1.1 AIM AND OBJECTIVES

The main aim for this project is:

- To compile all the necessary information into a book to enlighten people on the study of concave and convex functions.

The specific objectives are:

- To give a write up on the study of the basic properties, definitions, proofs of theorems and examples of convex functions.
- To compile other applications of concave and convex functions.

1.2 METHODOLOGY

Convex function is itself a mathematical tool. Most of our findings will be from literature from the library. It will also be from information gathered from the internet and consultations from people who have ideas on convex functions.

1.3 JUSTIFICATION

Although convex functions play an important role in Mathematics, its total understanding and application by people is not fully appreciated. Therefore this project seeks to give a concise note on convex functions and also to help people to appreciate convex functions and know its application.

CHAPTER TWO

LITERATURE REVIEW

2.1 CONVEX SETS

2.1.1 DEFINITIONS

A convex set is a set of elements from a vector space such that all the points on the straight line between any two points of the set are also contained in the set. If a and b are points in a vector space the points on the straight line between a and b are given by

$$x = \lambda a + (1 - \lambda) b \text{ for all } \lambda \text{ from 0 to 1}$$

The above definition can be restated as: A set S is convex if for any two points $a \text{ and } b$ belonging to S there are no points on the line between $a \text{ and } b$ that are not members of S.

Another restatement of the definition is: A set S is convex if there are no points a and b in S such that there is a point on the line between $a \text{ and } b$ that does not belong to S. The point of this restatement is to include the empty set within the definition of convexity. The definition also includes singleton sets where $a \text{ and } b$ have to be the same point and thus the line between $a \text{ and } b$ is the same point (Rawlins G.J.E. and Wood D, 1988).

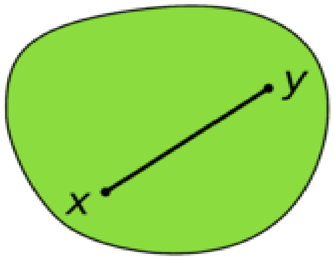


Figure 1 A convex set.

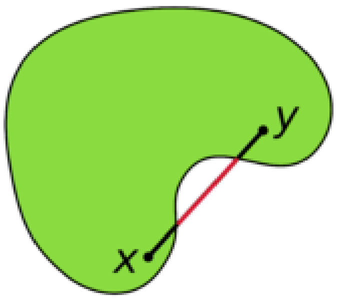


Figure 2 A non convex set

In figure 1, the line segment joining X and Y is contained in the set therefore it is a convex set. But in figure 2, part of the line segment is outside the set hence it's not a convex set.

For example, the two-dimensional set on the left of the following figure is convex, because the line segment joining every pair of points in the set lies entirely in the set. The set on the right is not convex, because the line segment joining the points *x* and *x*' does not lie entirely in the set.

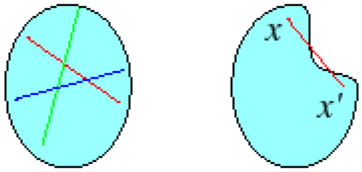


Figure 3 A convex and a non convex set

In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object. For example, a solid cube is convex, but anything that is hollow or has a dent in it, for example, a crescent shape, is not convex.

A subset C of a Euclidean space is convex if it contains the line segment connecting any two of its members. That is, if x and y are vectors in C and t is a number between 0 and 1, the vector $tx + (1- t)\, y$ is also in C. A linear combination with non-negative weights which sum to 1 is convex combination of elements of C; a set C is convex if it contains all convex combinations of its elements (Munkres, James, 1999).

Let C be a set in a real or complex vector space. C is said to be convex if, for all x and y in C and all t in the interval [0,1], the point

$$(1 - t)\, x + t\, y$$

is in C. In other words, every point on the line segment connecting x and y is in C. This implies that a convex set is connected.

The convex subsets of R (the set of real numbers) are simply the intervals of R. Some examples of convex subsets of Euclidean 2-space are regular polygons and bodies of constant width. Some examples of convex subsets of Euclidean 3-space are the Archimedean solids and the Platonic solids (Rawlins G.J.E. and Wood D).

2.1.2 THEOREM

The intersection of any two convex sets is a convex set (Soltan, Valeriu, 1984). With the inclusion of the empty set as a convex set then it is true that:

The proof of this theorem is by contradiction. Suppose for convex sets S and T there are elements $a\ and\ b$ such that a and b both belong to $S \cap T$, i.e., a belongs to S and T and b belongs to $S\ and\ T$ and there is a point c on the straight line between a and b that does not belong to $S \cap T$. This would mean that c does not belong to one of the sets $S\ or\ T$ or both. For whichever set c does not belong to this is a contradiction of that set's convexity, contrary to assumption. Thus no such $c\ and\ a\ and\ b$ can exist and $hence\ S \cap T$ is convex.

In simple terms, the definition of convex sets can be summarized as follows;

- A set S is convex if the line segment between any two points in S lies in S.
- A set S is convex if every point in the set can be seen by every other point, along an unobstructed means lying in the set.
- A set S is said to be convex if whenever it contains two points $a\ and\ b$, it also contains the line segment joining them (Soltan, Valeriu, 1984).

2.1.3 AFFINE SETS

A set S is an affine set if and only if $x, y \in S$ and $\alpha, \beta \in R$,where $\alpha+\beta=1, \alpha x+\beta y \in S$. Thus, S is affine if and only if the line passing through any two distinct points of S is itself a subset of S (Moon, T., 2008).

Examples of affine sets are planes, lines, single points

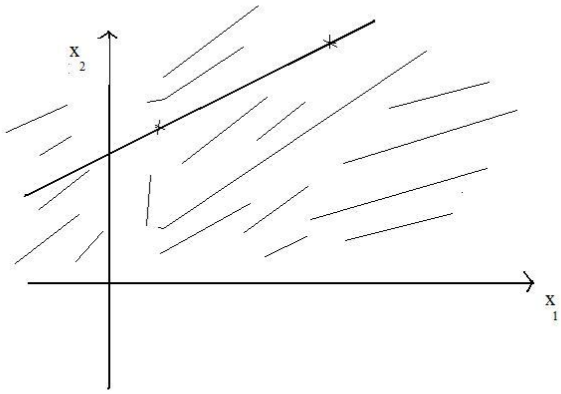
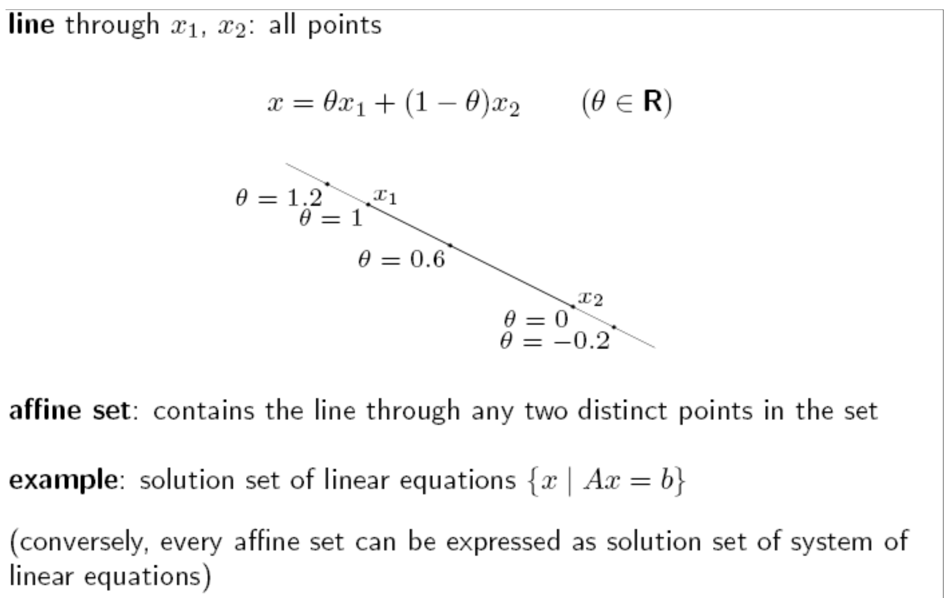


Figure 4 A non affine set

The above figure is not affine because the line passing through the two points extends outside the region.



2.1.4 COMBINATIONS

I. A linear combination of the vectors $\alpha_i \in R$.is any expression of the form $\sum \alpha_i x_i$ where $\alpha_i \in R$.

II. Affine combination of the vectors $x_{1,} x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\sum \alpha_i = 1$, $\alpha_i \in R$.

III. Conical combination of the vectors $x_{1,} x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\alpha_i \geq 0$, $\alpha_i \in R$..

IV. Convex combinations of the vectors $x_{1,} x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\sum \alpha_i = 1$, $\alpha_i \geq 0$, $\alpha_i \in R$.

2.1.5 HULLS

I. Linear hull $[L(S)]$, is the collection of all linear combinations of the vectors of S. i.e L(S) =$\{\sum \alpha_i x_i$, each $x_i \in S$,each $\alpha_i \in R.\}$

II. Affine hull $[Aff(S)]$, is the collection of all affine combinations of the vectors of S i.e Aff(S)=$\{\ \sum \alpha_i x_i,\ each\ x_i \in S, each\ \alpha_i \in R, \sum \alpha_i = 1.\}$

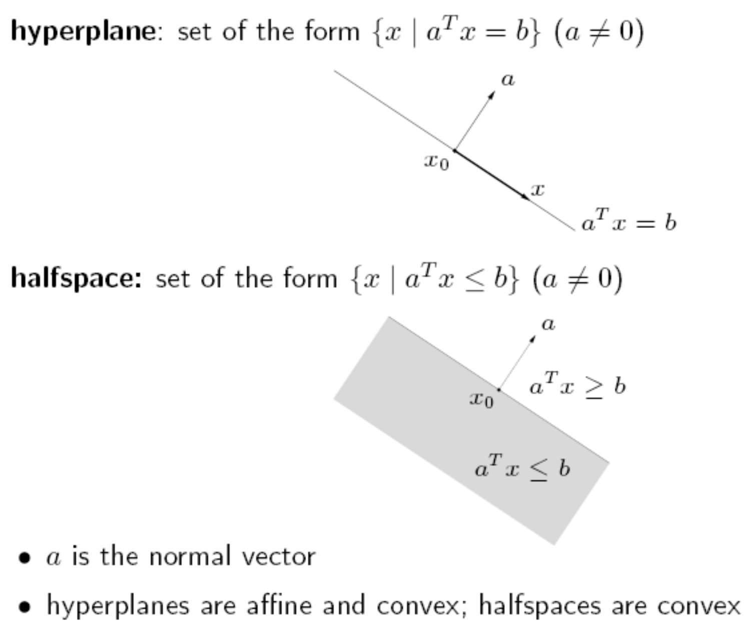
III. Conical hull [Coni(S)], is the collection of all conical combinations of the vectors of i.e Coni(S)=$\{\ \sum \alpha_i x_i\ each\ x_i \in S, each\ \alpha_i \in R, \alpha_i \geq\ 0.\}$

IV. Convex hull [Conv(S)], is the collection of all convex combinations of the vectors of S i.e Conv(S= $\{\sum \alpha_i x_i, each\ x_i \in S, each\ \alpha_i \in R, \alpha_i \geq 0,\ \sum \alpha_i = 1\}$

2.1.6 HYPERPLANE AND HALFSPACES

Let C denote a non zero vector in R^n,then the set $\{x \in R^n: c^T x = k\}$ is a Hyperplane in R. Thus in R^2,a hyper plane is the set of all points (x_1, x_2) such that $c_1x_1+c_2x_2=k$ where $(c_1,c_2)^T \neq 0$,that is, a hyperplane in R^2 is a straight line. In R^3, take c= $(c_1,c_2,c_3)^T$ then a hyperplane in R^3 is the set of all points $(x_1,x_2,x_3)^T$ such that $c_1x_1+c_2x_2+c_3x_3=k$. Thus a hyperplane in R^3 is a plane (Bertsekas, Dimitri, 2003).

Let H=$\{x \in R^n: c^T x = k\}$ denote a hyperplane in R^n. If R^n is divided by H into two parts$\{x \in R^n: c^T x \leq k\}$ and $\{x \in R^n: c^T x \geq k\}$. Each of these parts is called a closed half space. R^n can also be divided into three disjoint parts by the hyperplane H namely H itself $\{x \in R^n: c^T x = k\}$, $\{x \in R^n: c^T x < k\}$ and $\{x \in R^n: c^T x > k\}$. Each of these last two parts is called an open half (Bertsekas, Dimitri, 2003).



2.1.7 CONES

A set K is a cone if $\lambda x \in K$ for each $x \in K$ and $\lambda \geq 0$. A convex cone is a cone which is a convex set. A set K is a convex cone if it is convex and a cone, which means that for any $x_1, x_2 \in K$ and $\lambda_1, \lambda_2 \geq 0$ we have $\lambda_1 x_1 + \lambda_2 x_2 \in K$.

A point of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$ with $\lambda_1, \ldots, \lambda_k \geq 0$ or $[\sum \lambda_i x_i, where \lambda_i \geq 0]$ is called a conic combination of $x_1, \ldots, x_n$. If x_i are in a convex cone K, then every conic combination of x_i is in K. Conversely, a set K is a convex cone if and only if, it contains all conic combinations of its elements.

A finite cone is the sum of a finite number of rays.

1. The intersection of an arbitrary collection of convex cones is a convex cone.
2. A finite cone is a closed convex cone.

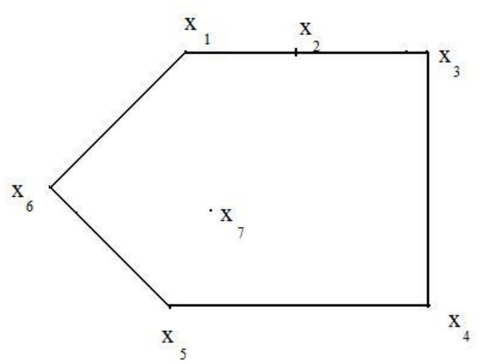
Geometrically, a convex cone with vertex the origin is any convex set that contains all the rays from the origin through each of the points in the set. The set

$S = \{(x_1, x_2)^T : x_1 = |x_2|\}$ is not a convex cone with vertex the origin since it is not a convex set. The set $T = \{(x_1, x_2)^T : x_1 \geq |x_2|$ is such a cone (Hiriart-Urruty *et al*, 2004).

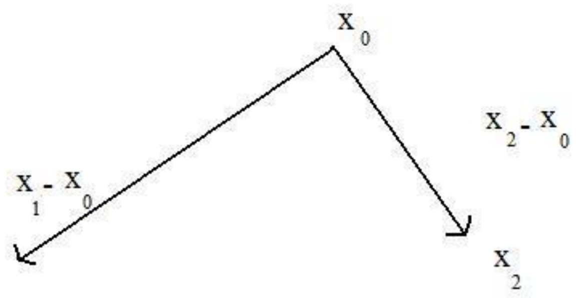
2.1.8 CONVEX POLYTOPE

Convex set play a very important role in mathematical programming. The convex hull of finitely many points is called a convex polytope. The set of vectors $\{x_0, x_1, \ldots, x_k\}$ is affinely independent if and only if the set$\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent (Hiriart-Urruty *et al*, 2004).

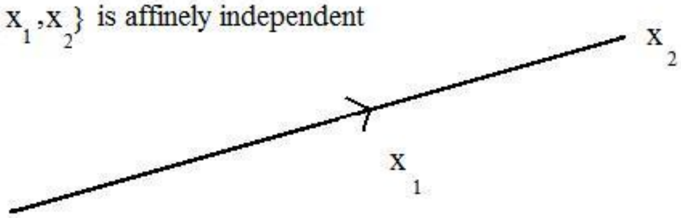
A convex polytope is called a k-dimensional simplex if and only if it is of $k + 1$ affinely independent vectors. A convex polytope is an enclosed figure. The diagram below is an example of a convex polytope:



$$S = conv\{x_1, x_2, \ldots, x_7\}$$



$\{x_0, x_1, x_2\}$ is affinely independent



$\{x_0, x_1, x_2\}$ is not affinely independent

2.1.9 SIMPLEX

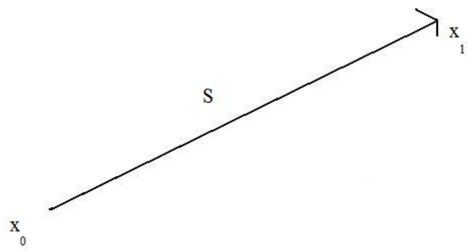
A simplex is an n-dimensional convex polytope having exactly $(n + 1)$ vertices.

A set in R^n which is a convex hull of $(n + 1)$ points in R^n is called n-simplex.

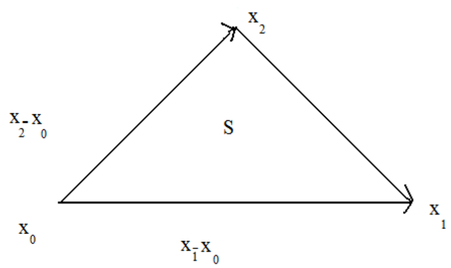
For instance;

A simplex in 0-dimension is a point.

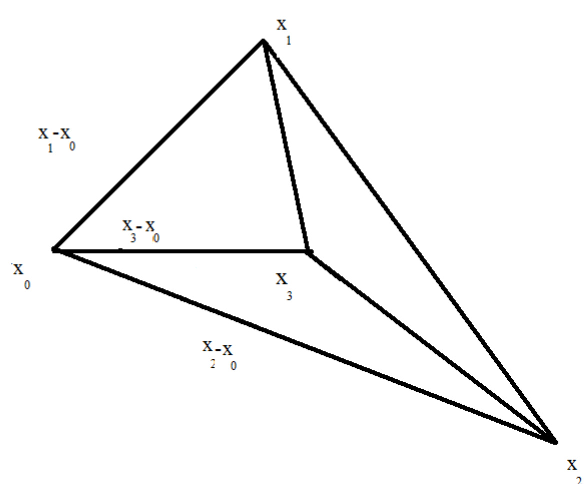
a simplex in 1-dimension is a line.



a simplex in 2-dimension is a triangle

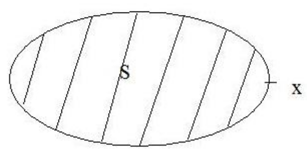


a simplex in 3-dimension is a tetrahedron

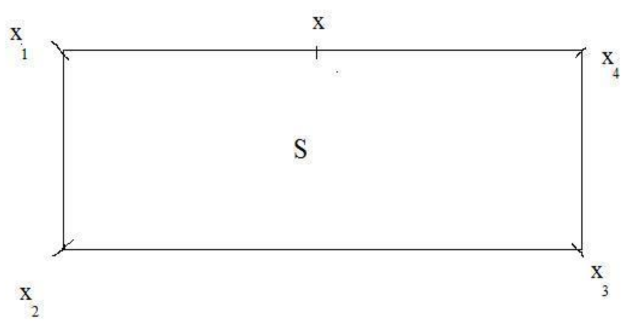


2.1.10 EXTREME POINTS

The extreme points of a figure are the points on the boundary of that figure. Examples can be seen in the figures below:



The extreme point of this figure is the point x lying on its boundary.



The extreme points of S are x_1, x_2, x_3, x_4 and x on the boundary.

Clearly, the extreme points of a convex set S are those points of S that do not lie on the interior of any line segment connecting any other pair of points of S(Luenberg D., 1984).

2.1.11 EXAMPLES OF CONVEX SETS

1. R^n
2. Point
3. Hyperplane or closed or open half space
4. Line
5. Line segment

2.2 CONCAVE SETS

2.2.1 DEFINITION

In mathematics, the notion of concave set is complementary to that of the convex set.

The following definitions are in use.

- A set is called concave if it is not convex.
- A set is called concave if its complement is convex.

2.3 CONVEX FUNCTION

2.3.1 DEFINITION

In mathematics, a real-valued function f defined on an interval (or on any convex subset of some vector space) is called convex, concave upwards, concave up or convex cup. If for any two points x and y in its domain C and any t in [0,1], we have

$$f\ (tx + (1-t)y)\ \leq\ tf(x) + (1-t)f(y)$$

In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.

Pictorially, a function is called 'convex' if the function lies below the straight line segment connecting two points, for any two points in the interval [0,1].

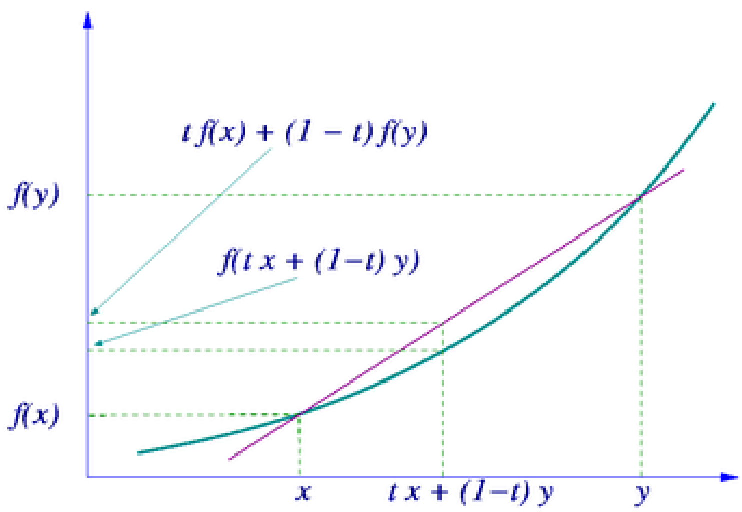


Figure 5 A convex function

A function is called strictly convex if

$$f(tx+(1-t)y) < tf(x)+(1-t)f(y)$$

for any $t \ in$ (0,1) $and\ x \neq y$.

Let f be a function of several variables, defined on a convex set S. We say that f is convex if the line segment joining any two points on the graph of f is never below the graph.

2.3.2 DEFINITION

Let f be a function of several variables defined on the convex set S. Then f is

- convex on the set S if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0,1)$ we have

  $$f\,((1-\lambda)x + \lambda x') \leq (1-\lambda)\, f\,(x) + \lambda\, f\,(x').$$

- strictly convex function is one that satisfies the definition for concavity with a strict inequality ($<$ rather than $\leq$) for all $x \neq x'$.

2.3.3 DEFINITION

- If a convex (i.e., concave upward) function has a "bottom", any point at the bottom is a minimal extremum. If a concave (i.e., concave downward) function has an "apex", any point at the apex is a maximal extremum.

- If $f(x)$ is twice-differentiable, then $f(x)$ is concave if and only if $f''(x)$ is non-positive. If its second derivative is negative then it is strictly concave, but the opposite is not true, as shown by $f(x) = -x^4$.

2.3.4 EXAMPLES OF CONVEX FUNCTIONS

- The function $f(x) = x^2 \, has \, f''(x) = 2 > 0$ at all points, so f is a (strictly) convex function.
- The absolute value function $f(x) = |x|$ is convex, even though it does not have a derivative at the point $x = 0$.
- The function $f(x) = |x| \, p \, for \, 1 \leq p$ is convex.
- The exponential function $f(x) = e^x$ is convex. More generally, the function $g(x) = e^{f(x)}$ is logarithmically convex if f is a convex function.
- The function f with domain [0,1] defined by $\mathrm{f}(0) = f(1) = 1, f(x) = 0$ for $0 < x < 1$ is convex; it is continuous on the open interval (0,1), but not continuous at $0 \, and \, 1$.
- The function x^3 has second derivative $6x$; thus it is convex on the set where $x \geq 0$ and concave on the set where $x \leq 0$.
- Every norm is a convex function, by the triangle inequality.

2.4 CONCAVE FUNCTIONS

2.4.1 DEFINITION

In mathematics, a concave function is the negative of a convex function. A concave function is also synonymously called concave downwards, concave down or convex cap (Rockafellar, R. T.,1970).

Formally, a real-valued function f defined on an interval (or on any convex set C of some vector space) is called concave, if for any two points $x \; and \; y$ in its domain C and any t in [0,1], we have

$$f\,(tx + (1-t)y) \;\geq\; tf(x) + (1-t)f(y)$$

Also, $f(x)$ is concave on $[a, b]$ if and only if the function $-f(x)$ is convex on $[a, b]$.

A function is called strictly concave if

$$f\,(tx + (1-t)y) > tf(x) + (1-t)f(y)$$

for any $t \; in$ (0,1) $and \; x \neq y$.

This definition according to Rockafellar, R. T. (1970), merely states that for every z between $x \; and \; y$, the point $(z, f(z)\,)$ on the graph of f is above the straight line joining the points $(x, f(x)\,)$ and $(y, f(y)\,)$.

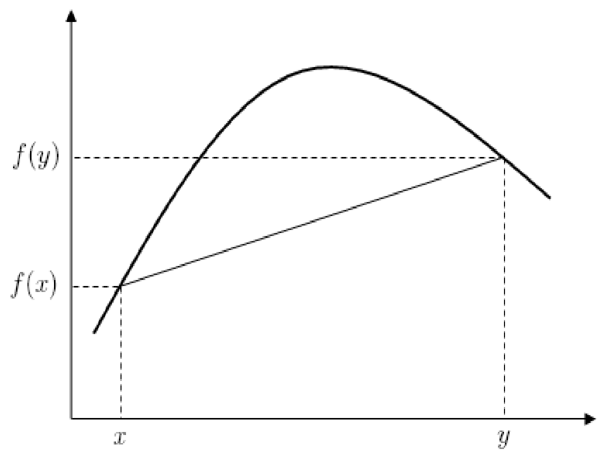


Figure 6 A concave function

A continuous function on C is concave if and only if

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

for any $x\ and\ y\ in\ C$.

A differentiable function f is concave on an interval if its derivative function f ′ is monotonically decreasing on that interval: a concave function has a decreasing slope. ("Decreasing" here means "non-increasing", rather than "strictly decreasing", and thus allows zero slopes.)

Let *f* be a function of many variables, defined on a convex set *S*. We say that *f* is concave if the line segment joining any two points on the graph of f is never above the graph.

A function f is said to be concave if $-f$ is convex (Rawlins G.J.E. and Wood D,1988).

2.4.2 DEFINITION

Let f be a function of many variables defined on the convex set *S*. Then f is

- concave on the set *S* if for all $x \in S, all\ x' \in S$, and all $\lambda \in (0,1)$ we have

$$f\,((1-\lambda)x + \lambda x') \geq (1-\lambda)\,f\,(x) + \lambda\,f\,(x').$$

Once again, a strictly concave function is one that satisfies the definition for concavity with a strict inequality (> rather than ≥) for all $x \neq x'$.

2.4.3 PROPOSITION

A function f of many variables defined on the convex set *S* is

- concave if and only if the set of points below its graph is convex:

$\{(x, y): x \in S\ and\ y \leq f\,(x)\}$ is convex

2.4.4 PROPERTIES

For a twice-differentiable function f, if the second derivative, $f\,''(x)$, is positive (or, if the acceleration is positive), then the graph is convex; if $f\,''(x)$ is negative, then the graph is concave. Points where concavity changes are inflection points. (Stephen Boyd and Lieven Vandenberghe 2004).

2.4.5 QUASI-CONCAVE

A function $f(x)$ is quasi-concave if

$$f(x) \geq f(y) \; implies \; f(\, tx + (1-t)y \,) \geq f(y)$$

- If $f(x)$ is a function of one variable and is single-peaked, then $f(x)$ is quasi-concave.
- If $f(x)$ is quasi-concave, then its upper level sets are convex sets. The level curves (isoquants, indifference curves) are convex to the origin (diminishing marginal rate of substitution).
- If $f(x)$ is quasi-concave, then the Hessian matrix is negative semi-definite subject to constraints. In particular, let H be the Hessian matrix of $f(x)$, and let f' be the vector of first derivatives of $f(x)$, then

$$x' \, H\,(x) \leq 0 \; for \; all \; f'\,(x) = 0$$

If $f(x)$ is quasi-concave, then the bordered Hessian matrix

```
        [ f_11  f_12  ...  f_1n  f_1 ]
        [ f_21  f_22  ...  f_2n  f_2 ]
    H = [ ........................... ]
        [ f_n1  f_n2  ...  f_nn  f_n ]
        [ f_1   f_2   ...  f_n   0   ]
```

have border-preserving principle minors determinants of order k that alternate in sign.

i.e., the $3x3$ (including the border) determinant is positive, the 4 x 4 determinant is negative, and so on. Note that quasi-concavity does not imply that $f_{11} \leq 0$.

If $f(x)$ is quasi-concave, then $-f(x)$ is quasi-convex. The lower level sets of a quasi-convex function are convex (Press, W. H *et al*., 1995).

Relationship between concavity and quasi-concavity:

- All concave functions are quasi-concave.
- Any monotonic transformation of concave function is quasi-concave.
- Quasi-concave functions are not necessarily concave.

A set S is convex if for any two elements $x \text{ and } y$ that belongs to S, the element $tx + (1-t)\,y$ also belongs to S (t *is between* 0 *and* 1).

2.5 CONCAVE AND CONVEX FUNCTIONS OF SINGLE VARIABLES

2.5.1 GENERAL DEFINITIONS

The twin notions of concavity and convexity are used widely in economic theory, and are also central to optimization theory. A function of a single variable is concave if every line segment joining two points on its graph does not lie above the graph at any point.

Symmetrically, a function of a single variable is convex if every line segment joining two points on its graph does not lie below the graph at any point. These concepts are illustrated in the following figure.

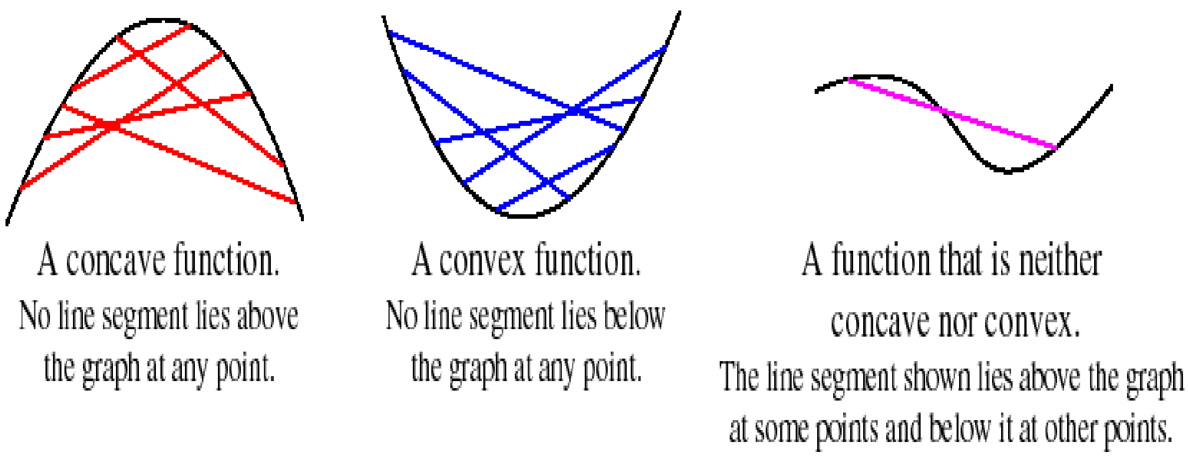


Figure 7 Function of a single variable

Here is a precise definition.

Note that a function may be both concave and convex. Let f be such a function. Then for all values of a and b we have

$$f\,((1-\lambda)a\,+\,\lambda b)\;\geq\;(1\,-\,\lambda)\,f\,(a)\,+\,\lambda\,f\,(b)\;for\;all\;\lambda\,\in\,(0,1)\;and$$

$$f\,((1-\lambda)a\,+\,\lambda b)\;\leq\;(1\,-\,\lambda)\,f\,(a)\,+\,\lambda\,f\,(b)\;for\;all\;\lambda\,\in\,(0,1).$$

Equivalently, for all values of a and b we have

$f\,((1-\lambda)a\ +\ \lambda b)\ =\ (1\ -\ \lambda)\,f\,(a)\ +\ \lambda\,f\,(b)$ for all λ ∈ (0, 1). That is, a function is both concave and convex if and only if it is linear, taking the form $f(x) = ax + b$ for all x, for some constants a and b.

Proof

$$f(\alpha x' + (1-\alpha)x") = a[\alpha x' + (1-\alpha)x"] + b$$

$$= [ax' + b] + (1-\alpha)(ax" + b)$$

$$= \alpha f(x') + (1-\alpha) f(x")$$

Since both definitions of convex and concave functions are satisfied with equality, $f(x) = ax+b$ is both a convex and a concave function.

2.5.2 CONVEXITY OF FUNCTIONS OF SEVERAL VARIABLES

To determine whether a twice-differentiable function of many variables is concave or convex, we need to examine all its second partial derivatives. We call the matrix of all the second partial derivatives the Hessian of the function.

2.5.3 DEFINITION

Let f be a twice differentiable function of n variables. The Hessian of f at x is

H(x) = $f_{11}''(x)$ $f_{12}''(x)$ ... $f_{1n}''(x)$

$$\begin{bmatrix} f_{21}''(x) & f_{22}''(x) & \ldots & f_{2n}''(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_{n1}''(x) & f_{n2}''(x) & \ldots & f_{nn}''(x) \end{bmatrix}$$

2.5.4 DEFINITION

1. The hessian of $f(x_1, x_2, \ldots x_n)$ is the $n \times n$ matrix whose ij^{th} entry is $\frac{\partial^2}{\partial x_i \partial y_i}$

   Example

   Let $H(x_1, x_2, \ldots x_n)$ denote the value of the Hessian at (x_1, x_2) if $f(x_1, x_2) = x_1^3 + 2x_1x_2 + x_2^2$ then $H(x_1, x_2) = \begin{bmatrix} 6x_1 & 2 \\ 2 & 2 \end{bmatrix}$

2. An i^{th} principal minor of an $n \times n$ matrix is the determinant of any $i \times i$ matrix obtained by deleting $n - i$ rows and the corresponding$(n - i)$ columns of the matrix.

   Example

   The matrix$\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$ has first principal minors as -2 and -4 and the 2^{nd} principal minor is $-2(-4) - (-1)(-1) = 7$.

3. The K^{th} leading principal minor of an $n \times n$ matrix is the determinant of the $K \times K$ matrix obtained by deleting the last $(n-k)$ rows and columns of the matrix.

Example

Let $H_k(x_{1,}x_2, \dots, x_n)$ be K^{th} leading principal minor of the Hessian matrix evaluated at the point $(x_1, x_2, \dots, x_n)$. Then if $f(x_1, x_2) = x_1^3 + 2x_1x_2 + x_2^2$ then $H_1(x_1, x_2) = 6x_1$

$H_2(x_1, x_2) = 6x_1(2) - 2(2) = 12x_1 - 4.$

2.5.5 THEOREM

- Suppose $f(x_1, x_2, \dots, x_n)$ has continuous second-order partial derivatives for each point $(x_1, x_2, \dots, x_n) \in S$. then $f(x_1, x_2, \dots, x_n)$ is a convex function S if and onl if for each $x \in S$, all principal minors of$H$ are non$-$negative.
- Suppose $f(x_1, x_2, \dots, x_n)$ has continuous second-order partial derivatives for each point$x = (x_1, x_2, \dots, x_n) \in S$. Then $f(x_1, x_2, \dots, x_n)$ is a concave function on S if and only if for each$x \in S$ and $K = 1, 2, \dots, n$, all non-zero principal minors have the same sign as$(-1)^K$.

B applying theorem (1) and (2) above, the Hessian matrix can be used to determine whether $f(x_1, x_2, \dots, x_n)$a convex is or a concave function on a convex set $S \subset R^n$.

Examples

I. Show that $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ is a convex function on $S = R^2$.

Solution

$H(x_1, x_2) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

1st principal minors of the Hessian are the diagonal entries and both are ≥0. The 2nd principal minor is $2(2) - 2(2) = 0 \geq 0$. Since for any point, all principal minors of H are non-negative, theorem (1) shows that $f(x_1, x_2) = -x_1^2 - x_1x_2 - 2x_2^2$

Is a concave function on R^2.

II. Show that for $S = R^2$, $f(x_1, x_2) = x_1^2 - 3x_2x_2 + 2x_2^2$ is not a convex or a concave function.

Solution

$$H(x_1, x_2) = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

1st principal minors of the Hessian are the diagonal entries and both of them ≤ 0.

The 2nd principal minor is $2(2) - (-3)(-3) = 5 \geq 0$. since the 1st principal minor is negative and the 2nd principal minor is positive, the function is neither convex nor concave.

Consider the function $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3$ defined on the set of all triples of numbers. Its first partials are

$f'_1(x_1, x_2, x_3) = 2x_1 + 2x_2 + 2x_3$

$f'_2(x_1, x_2, x_3) = 4x_2 + 2x_1$

$f'_3(x_1, x_2, x_3) = 6x_3 + 2x_1$.

So its Hessian is

$$\begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = Z \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

The leading principal minors of the Hessian are $2 > 0, 4 > 0, and\ 6 > 0$. So the Hessian is positive definite, and f is strictly convex.

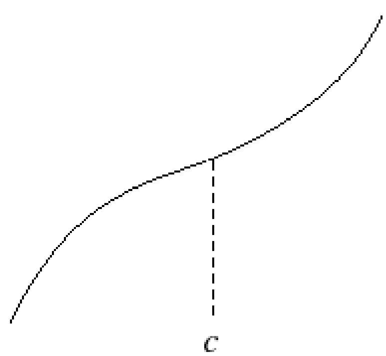
2.5.6 DEFINITION

A point at which a twice-differentiable function changes from being convex to concave, or vice versa, is an inflection point.

c is an inflection point of a twice-differentiable function f of a single variable if for some values of a and b with $a < c < b$ we have

- either $f''(x) \geq 0$ if $a < x < c$ and $f''(x) \leq 0$ if $c < x < b$
- or $f''(x) \leq 0$ if $a < x < c$ and $f''(x) \geq 0$ if $c < x < b$.

An example of an inflection point is shown in the following figure.



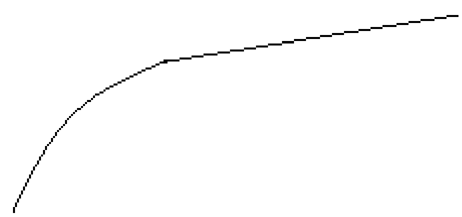
2.5.7 PROPOSITION

- If c is an inflection point of f then $f''(c) = 0$.
- If $f''(c) = 0$ and f'' changes sign at c then c is an inflection point of f.

Note, however, that f'' does not have to change sign at c for c to be an inflection point of f. For example, every point is an inflection point of a linear function.

2.5.8 STRICT CONVEXITY AND CONCAVITY

The inequalities in the definition of concave and convex functions are weak: such functions may have linear parts, as in the following figure.



a concave, but not strictly concave, function

A concave function that has no linear parts is said to be strictly concave.

2.5.9 DEFINITION

The function f of a single variable defined on the interval I is

- strictly concave if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have $f((1-\lambda)a + \lambda b) > (1 - \lambda) f(a) + \lambda f(b)$.
- strictly convex if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have

$$f((1-\lambda)a + \lambda b) < (1 - \lambda) f(a) + \lambda f(b).$$

An earlier result states that if f is twice differentiable then f is concave on $[a, b]$ if and only if $f''(x) \leq 0$ for all $x \in (a, b)$.

Does this result have an analogue for *strictly* concave functions? Not exactly. If $f''(x) < 0$ for all $x \in (a,b)$ then f is strictly concave on $[a, b]$, but the converse is not true: if f is strictly concave then its second derivative is not necessarily negative at all points. (Consider the function $f(x) = -x^4$ It is concave, but its second derivative at 0 is zero, not negative) That is, f is strictly concave on $[a, b]$ if $f''(x) < 0$ for all $x \in (a, b)$, but if f is strictly concave on $[a, b]$ then $f''(x)$ is not necessarily negative for all $x \in (a, b)$.
(Analogous observations apply to the case of convex and strictly convex functions, with the conditions $f''(x) \geq 0$ and $f''(x) > 0$ replacing the conditions $f''(x) \leq 0$ and $f''(x) < 0$)

CHAPTER 3

3.1 APPLICATIONS OF CONVEX FUNCTION AND CONCAVE FUNCTIONS

As a matter of fact, we experience convexity all the time and in many ways. The most prosaic example is our upright position, which is secured as long as the vertical projection of our center of gravity lies inside the convex envelope of our feet. Also, convexity has a great impact on our everyday life through numerous applications in industry, business, medicine, and art. So do the problems of optimum allocation of resources and equilibrium of non cooperative games. We shall limit ourselves to just a few of them.

Economists often assume that a firm's production function is increasing and concave. An example of such a function for a firm that uses a single input is shown in the next figure. The fact that such a production function is increasing means that more input generates more output. The fact that it is concave means that the increase in output generated by a one-unit increase in the input is smaller when output is large than when it is small. That is, there are "diminishing returns" to the input, or, given that the firm uses a single input, "diminishing returns to scale". For some (but not all) production processes, this property seems reasonable.

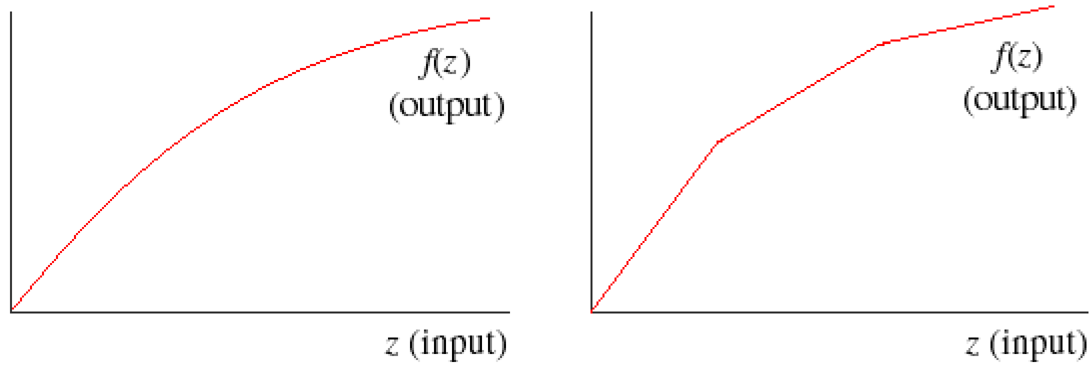


Figure 8 Two concave production functions

The notions of concavity and convexity are important in optimization theory because, as we shall see, the first-order conditions are sufficient (as well as necessary) for a maximizer of a concave function and for a minimizer of a convex function. (Precisely, every point at which the derivative of a concave differentiable function is zero is a maximizer of the function, and every point at which the derivative of a convex differentiable function is zero is a minimizer of the function.)

Three functions of importance to an economist or a manufacturer

$C(x)$ =total cost of producing x units of a product during some time period

$R(x)$ = total revenue received from selling x units of the product during the time period

$P(x)$ =total profit obtained by selling x units of the product during the time period

These are called, respectively, the cost function, revenue function, and profit function. If all units produced are sold, then these are related by

$$P(x) = R(x) - C(x)$$

That is profit = revenue−cost

The total cost *C(x)* of producing x units can be expressed as a sum

$$C(x) = a + M(x)$$

Where a is a constant, called overhead, and $M(x)$ is a function representing

manufacturing cost. The overhead, which includes such fixed costs as rent and insurance, does not depend on x; it must be paid even if nothing is produced. On the other hand, the manufacturing cost M(x), which includes such items as cost of materials and labour, depends on the number of items manufactured. It is shown in economics that with suitable simplifying assumptions, $M(x)$ can be expressed in the form

$$M(x) = bx + cx^2$$

Where b and c are constants. Substituting this in the previous equation yields

$$C(x) = a + bx + cx^2$$

If a manufacturing firm can sell all the items it produces for p dollars apiece, then its total revenue *R(x)* (in dollars) will be

$$R(x) = px$$

And its total profit *P(x)* (in dollars) will be

$$P(x) = [total\ revenue] - [total\ cost] = R(x) - C(x) = px - C(x)$$

Thus, the cost function is given by

$$P(x) = px - (a + bx + cx^2)$$

Depending on such factors as number of employees, amount of machinery available, economic conditions, and competition, there will be some upper limit *l* on the number of items a manufacturer is capable of producing and selling. Thus, during a fixed time period the variable *x* in the previous equation will satisfy

$$0 \leq x \leq l$$

By determining the value or values of *x* in [0,*l*] that maximize the previous equation, the firm can determine how many units of its product must be manufactured and sold to yield the greatest profit.

This is illustrated in the example below.

Examples

1. A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for *x* units is

   $$C(x) = 500{,}000 + 80x + 0.003x^2$$

   And if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit?

   Solution.

Since the total revenue for selling x units is $R(x) = 200x$, the profit $P(x)$ on x units will be $P(x) = R(x) - C(x) = 200x - (500{,}000 + 80x + 0.003x^2)$

Since the production capacity is at most 30,000 units, x must lie in the interval [0,30,000].

$$dP/dx = 200 - (80 + 0.006x) = 120 - 0.006x$$

$$setting\ dP/dx = 0\ gives\ 120 - 0.006x = 0\ or\ x = 20{,}000$$

since this critical point lies in the interval [0,30,000], the maximum profit must occur at one of the points

$x = 0$, $x = 20{,}000$, or $x = 30{,}000$

substituting these values yields the table below, which tells us that the maximum profit

$P = \$700{,}000$ occurs when $x = 20{,}000$ units are manufactured and sold in the specified time.

x	0	20,000	30,000
$P(x)$	−500,000	700,000	400,000

2. A competitive firm receives the price $p > 0$ for each unit of its output, and pays the price $w > 0$ for each unit of its single input. Its output from using x units of the

variable input is $f(x) = x^{1/4}$. Is this production function concave? Is the firm's profit concave in x?

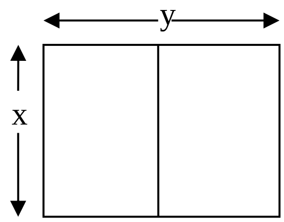
Solution

The function f is twice-differentiable for $x > 0$. We have $f'(x) = (1/4)x^{-3/4}$ and $f''(x) = -(3/16)x^{-7/4} < 0$ for all x, so f is concave for $x > 0$. It is continuous, so it is concave for all $x \geq 0$. The firm's profit, $p f(x) - wx$, is thus the sum of two concave functions, and is hence concave.

Often in life, we are faced with the problem of finding the best way to do something. For example, a farmer wants to choose the mix of crops that is likely to produce the largest profit. A doctor wishes to select the smallest dosage of drug that will cure a certain disease. A manufacturer would like to minimize the cost of distributing its products. Most at times problems of this nature can be formulated so it involves maximizing or minimizing a function over a specified set. Convex and Concave functions are best used for problems of this nature.

Suppose then that we have a function f and a domain S. The first thing to do is to decide whether f has a maximum value or a minimum value on S. Secondly, assuming that such values exist, we are interested in knowing where on S they are attained. Finally, we wish to determine the maximum and the minimum values.

1. Farmer Brown has 100 meters of wire fence with which he plans to build two identical adjacent pens, as shown in the diagram below. What are the dimensions of the total enclosure for which its area is a maximum?



Solution

let x be the width and y the length of the total enclosure, both in meters. Because there are 100 meters of fence, 3x+2y=100 that is y=50−3/2x. The total area, A is given by $A = xy = 50x - \frac{3}{2}x^2$. Since there must be three sides of length x, we see that

$0 \leq x \leq \frac{100}{3}$. Thus, our problem is to maximize A on $[0, \frac{100}{3}]$. Now,

$\frac{dA}{dx} = 50 - 3x$ when we set $50 - 3x$ equal to 0 and solve, we get $x = \frac{50}{3}$ as a stationary

point. Thus, there are three critical points $0, \frac{50}{3}$ and $\frac{100}{3}$. The two endpoints 0 and $\frac{100}{3}$

give A=0. While $x = \frac{50}{3}$ yields y=50−$\frac{3}{2}(\frac{50}{3}) = 25$meters.

2. Convexity can be used to find local extreme values.

   For example find the local extreme values of the function $f(x) = x^2 - 6x + 5$

   on(−∞,∞).

Solution;

The polynomial function f is continuous everywhere, and its derivative, $f'(x) = 2x - 6$, exists for all x. Thus, the only critical point for f is the single solution of $f'(x) = 0$, namely, $x = 3$. Since $f'(x) = 2(x - 3) < 0 \; for \; x < 3$, f is decreasing on (−∞,3];and because $2(x - 3) > 0$ for x>0, f is increasing on [3,∞). Therefore, by the first derivative test, $f(3) = -3$ is a local minimum value of f. Since 3 is the only critical number, there are no other extreme values. The graph of f is shown in the figure below. Note that f(3) is actually the global minimum value in this case.

3. For $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$, use the second derivative test to identify local extrema.

Solution:

$f'(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$

$f''(x) = 2x - 2$

The critical points are -1 and $3 (f'(-1) = f'(3) = 0)$.

Since $f''(-1) = -4$ and $f''(3) = 4$, we conclude by the second derivative test that $f(-1)$ is a local maximum value and that f(3) is a local minimum value.

4. Finding extrema on open intervals

Find(if possible) the minimum and maximum values of $f(x) = x^4 - 4x$ on $(-\infty,\infty)$.

Solution:

$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1)$

since $x^2 + x + 1 = 0$ has no real solutions(quadratic formula), there is only one critical point, namely, $x = 1$. For $x < 1, f'(x) < 0$, whereas for $x > 1, f'(x) > 0$. We conclude that $f(1) = -3$ is a local minimum value for f; and since f is decreasing on the left of 1 and increasing on the right of 1, it must actually be the minimum value of f. This implies that f cannot have a maximum value. The graph is shown below.

5. Practical problem

A rectangular beam is to be cut from a log with circular cross section. If the strength of the beam is proportional to the product of its width and the square of its depth, find the dimensions of the cross section that give the strongest beam.

Solution: Denote the diameter of the log by a(constant) and the width and depth by w and d, respectively(as in the figure). We want to maximize S, the strength of the beam. From the conditions given in the problem,

$$S = kwd^2$$

Where k is a constant of proportionality. The strength S depends on the two variables w and d, but there is a simple relationship between them.

$$d^2 + w^2 = a^2$$

when we solve this equation for d^2 and substitute in the formula for S in terms of the single variable w.

$$S = kw(a^2 - w^2) = ka^2w - kw^3$$

We consider the allowable values for w to be 0<w<a, an open interval. To find the critical points we calculate dS/dw, set it equal to 0, and solve for w.

$$dS/dw = ka^2 - 3kw^2 = k\,(a^2 - 3w^2)$$

$$k\,(a^2 - 3w^2) = 0$$

$$3w^2 = a^2$$

$$w^2 = a^2/3$$

$w = a/\sqrt{3}$

$Since\ a/\sqrt{3}$ is the only critical point in (0,a), it is likely that it gives the maximum S. when we substitute $w = a/\sqrt{3}\ in\ \ d^2 + w^2 = a^2$ we learn that d=√2a/√3. The desired dimensions are w= a/√3 and d=√2a/√3 and d=√2w.

3.2 EXERCISES

1. Let S= $\{(1\ 0\ 0)^T, (0\ 1\ 0)^T\}$.Determine geometrically:

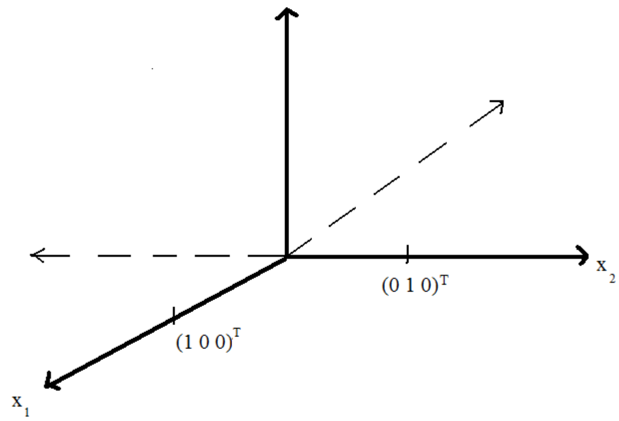
(i) L(S), (ii) aff(S), (iii)coni(S), (iv)conv(S)

Solution

(i) L(S) $=\{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha,\beta\in R\}=\{(\alpha\ \beta\ 0\ )^T:\alpha,\beta\in R\}$

$=\{(\alpha\ \beta\ 0)^T:\alpha\ \beta\in R\}$

$=x_1x_2$ plane

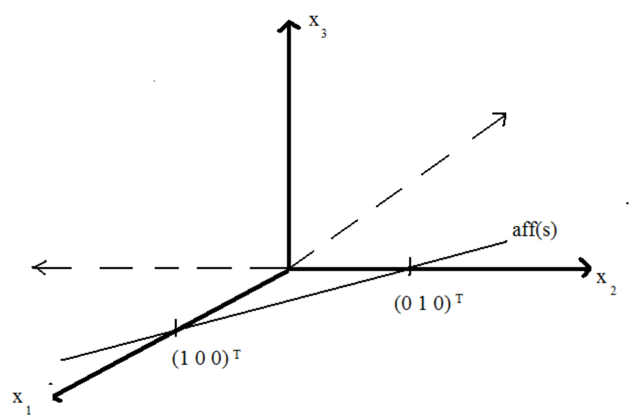


L(S)=L{$(1\ 0\ 0)^T$,$(0\ 10)^T$} is the x_1x_2 plane

(ii) Aff(S) = $\{\alpha(1\ 0\ 0)^T + \beta(0\ 1\ 0)^T : \alpha,\beta \in R, \alpha+\beta=1\}$

$= \{(\alpha\ \beta\ 0)^T : \alpha,\beta \in R, \beta=1-\alpha\}$

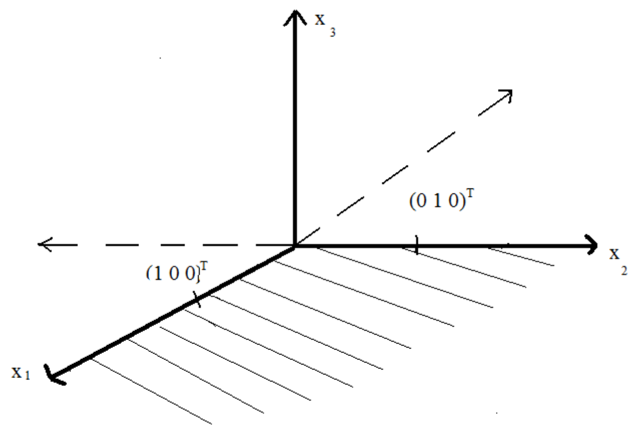
$=\{(\alpha, 1-\alpha, 0)^T : \alpha \in R\}$



Aff(S) is the line passing through $(1\ 0\ 0)^T$ and $(0\ 1\ 0)^T$.

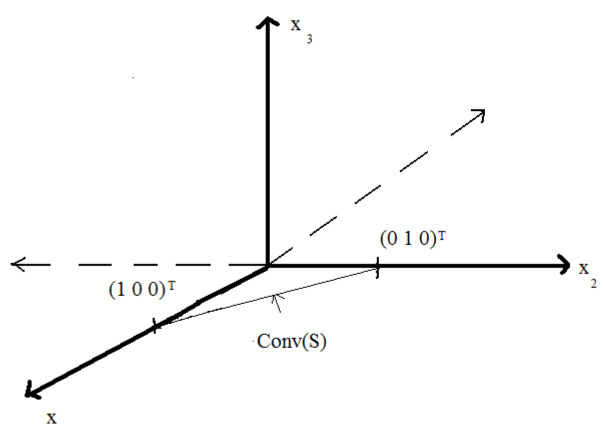
(iii) $Coni(S) = \{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha\ \beta\in R, \alpha \geq 0, \beta \geq 0\}$

$=\{(\alpha\ \beta\ 0)^T:\alpha \geq 0, \beta \geq 0\}$



(iv) $Conv(S) = \{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha,\beta\in R, \alpha \geq 0, \beta \geq 0, \alpha+\beta=1\}$

$=\{(\alpha\ 1\text{-}\alpha\ 0)^T:\alpha\in R, 1\geq\alpha\geq 0\}$

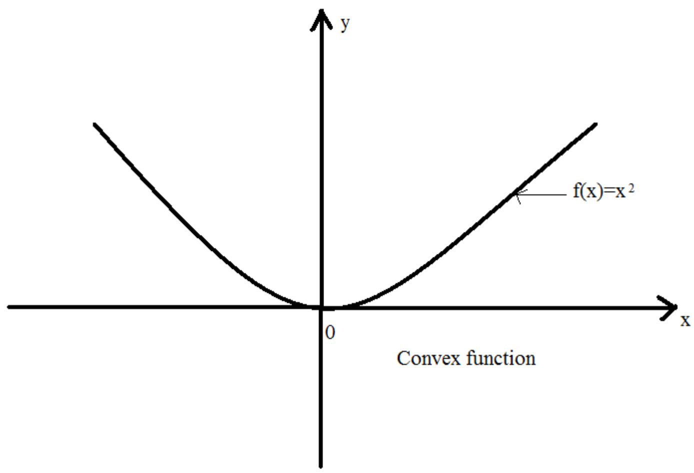


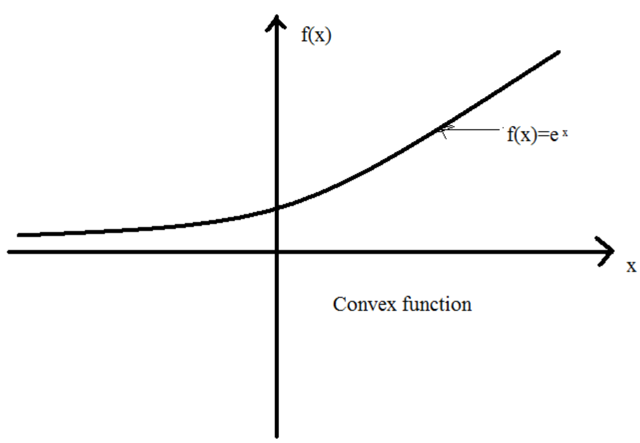
This is a straight line segment that connects the vectors $(1\ 0\ 0)^T$ and $(0\ 1\ 0)^T$

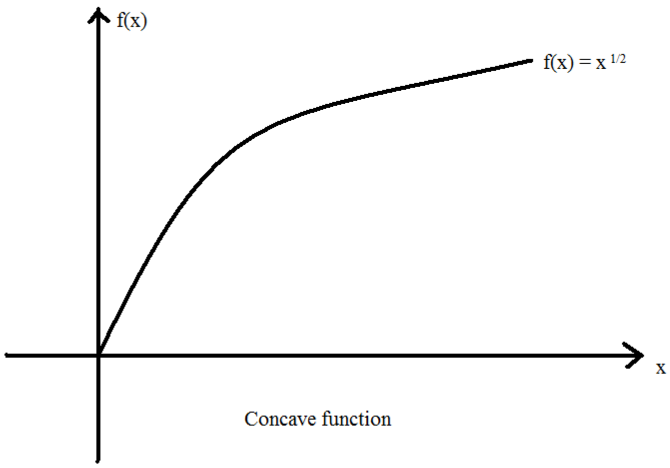
2. For $x \geq 0$, $f(x) = x^2$ and $f(x) = e^x$ are convex functions and $f(x) = x^{1/2}$ is a convex function.

Solution

These facts are evident in the figures below:







3. Show that a linear function of the form $f(x) = ax+b$ is both a convex and a concave function.

Proof

$$f(\alpha x'+(1-\alpha)x'')=a[\alpha x' + (1-\alpha)x'']+b$$

$$=[ax' +b] +(1-\alpha)(ax''+b)$$

$$=\alpha f(x')+(1-\alpha)f(x'')$$

Since both definitions of convex and concave functions are satisfied with equality, $f(x) = ax+b$ is both a convex and a concave function.

3.2.1 EXERCISES ON CONVEXITY AND CONCAVITY FOR FUNCTIONS OF SEVERAL VARIABLE

4.(a) By drawing diagrams, determine which of the following sets is convex.

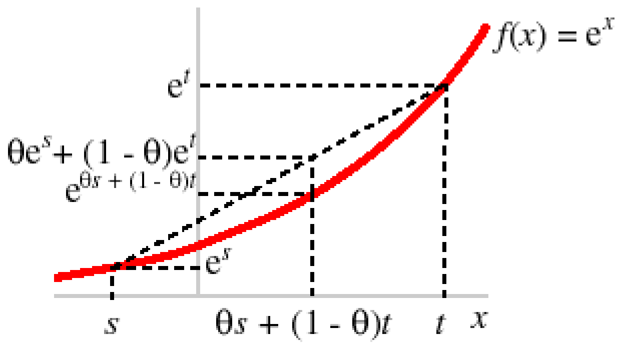
a. $\{(x, y): y = e^x\}$.

b. $\{(x, y): y \geq e^x\}$.

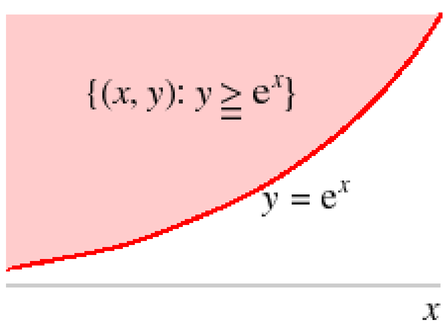
c. $\{(x, y): xy \geq 1, x > 0, y > 0\}$.

Solution

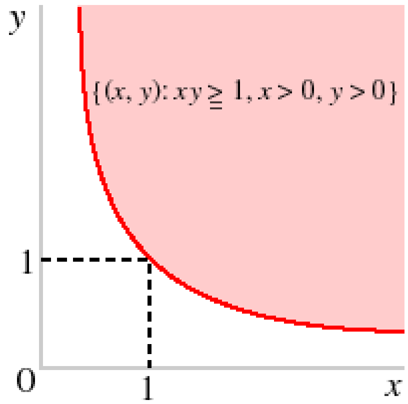
a. Not convex, because $e^{\theta x+(1-\theta)u} \neq \theta e^x + (1-\theta)e^u$, as illustrated in the following figure.



b. Convex, because $e^{\theta x+(1-\theta)u} < \theta e^x + (1-\theta)e^u$ (see the following figure).



c. Convex, because if $xy \geq 1$ and $uv \geq 1$ then $(\theta x + (1-\theta)u)(\theta y + (1-\theta)v) \geq 1$ (see figure).



5. Show that the intersection of two convex sets is convex.

Solution

Let A and B be convex sets. Let $x \in A \cap B$ and $x' \in A \cap B$. We need to show that $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$. Since $x \in A$, $x' \in A$, and A is convex we have $(1 - \lambda)x + \lambda x' \in A$ for all $\lambda \in [0, 1]$. Similarly $(1 - \lambda)x + \lambda x' \in B$ for all $\lambda \in [0, 1]$. Hence $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$.

6. The function C of many variables and the function D of a single variable are both convex. Define the function f by $f(x,k) = C(x) + D(k)$. Show that f is a convex function (without assuming that C and D are differentiable.

Solution

We have

$$f((1-\lambda)(x,k) + \lambda(x',k')) = C((1-\lambda)x + \lambda x') + D((1-\lambda)k + \lambda k')$$

$$\leq (1-\lambda)C(x) + \lambda C(x') + (1-\lambda)D(k) + \lambda D(k')$$

7. Let $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1 - 2x_2 + 1$. Is f convex, concave, or neither?

Solution

The Hessian matrix of f is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

3.2.2 EXERCISES ON CONCAVE AND CONVEX FUNCTIONS OF A SINGLE VARIABLE

8. Show that the function f is convex if and only if the function $-f$ is concave. [Do not assume that the function f is differentiable. The value of the function $-f$ at any point x is $-f(x)$.]

Solution

1. First suppose that the function f is convex. Then for all values of a and b with $a \leq b$ we have

   $f((1-\lambda)a + \lambda b) \leq (1-\lambda) f(a) + \lambda f(b)$.

   Multiply both sides of this equation by -1 (which changes the inequality):

$-f((1-\lambda)a + \lambda b) \geq -[(1-\lambda)f(a) + \lambda f(b)]$,

or

$-f((1-\lambda)a + \lambda b) \geq (1-\lambda)(-f(a)) + \lambda(-f(b))$.

Thus $-f$ is concave.

Now suppose that the function $-f$ is concave. Then

$$-f((1-\lambda)a + \lambda b) \geq (1-\lambda)(-f(a)) + \lambda(-f(b)).$$

Multiplying both sides of this equation by -1, gives

$f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b)$,

so that f is convex.

9. The functions f and g are both concave functions of a single variable. Neither function is necessarily differentiable.

a. Is the function h defined by $h(x) = f(x) + g(x)$ necessarily concave, necessarily convex, or not necessarily either?
b. Is the function h defined by $h(x) = -f(x)$ necessarily concave, necessarily convex, or not necessarily either?
c. Is the function $h(x) = f(x)g(x)$ necessarily concave, necessarily convex, or not necessarily either?

Solution

a. We have

$$h(\alpha x + (1-\alpha)y) = f(\alpha x + (1-\alpha)y) + g(\alpha x + (1-a)y)$$

$$\geq \alpha f(x) + (1-\alpha) f(y) + \alpha g(x) + (1-\alpha)g(y)$$

(using the concavity of f and of g)

$$= \alpha(f(x) + g(x)) + (1-\alpha)(f(y) + g(y))$$

$$= \alpha h(x) + (1-a)h(y).$$

b. Thus h is necessarily concave.

c. Since f is concave, we have

$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$ for all x, y, and α.

Hence

$-f(\alpha x + (1-\alpha)y) \leq \alpha(-f(x)) + (1-\alpha)(-f(y))$ for all x, y, and α,so that $-f$ is convex.

d. The function h is neither necessarily concave nor necessarily convex. If $f(x) = x$ and $g(x) = x$ then both f and g are concave, but h is convex and not concave. Thus h is not necessarily concave. If $f(x) = x$ and $g(x) = -x$ then both f and g are concave, and h is strictly concave, and hence not convex. Thus h is not necessarily convex.

10. The function $f(x)$ is concave, but not necessarily differentiable. Find the values of the constants a and b for which the function $a f(x) + b$ is concave. (Give a complete argument; no credit for an argument that applies only if f is differentiable.)

Solution

Let $g(x) = a\,f(x) + b$. Because f is concave we have

$$f((1-\alpha)x + \alpha y) \geq (1-\alpha)\,f(x) + \alpha\,f(y) \text{ for all } x, y, \text{ and } \alpha \in [0,1].$$

Now,

$$g((1-\alpha)x + \alpha y) = a\,f((1-\alpha)x + \alpha y) + b$$

and

$$(1-\alpha)g(x) + \alpha g(y) = a[(1-\alpha)\,f(x) + \alpha\,f(y)] + b.$$

Thus

$$g((1-\alpha)x + \alpha y) \geq (1-\alpha)g(x) + \alpha g(y) \text{ for all } x, y, \text{ and } \alpha \in [0,1]$$

if and only if

$$a\,f((1-\alpha)x + \alpha y) \geq a[(1-\alpha)\,f(x) + \alpha\,f(y)],$$

or if and only if $a \geq 0$ (using the concavity of f).

11. The function g of a single variable is defined by $g(x) = f(ax + b)$, where f is a concave function of a single variable that is not necessarily differentiable, and a and b are constants with $a \neq 0$. (These constants may be positive or negative.) Either show that the function g is concave, or show that it is not necessarily concave. [Your argument must apply to the case in which f is not necessarily differentiable.]

Solution We have

$$
\begin{aligned}
g(\alpha x_1 + (1-\alpha)x_2) &= f(a(\alpha x_1 + (1-\alpha)x_2) + b) \\
&= f(\alpha(ax_1 + b) + (1-\alpha)(ax_2 + b)) \\
&\geq \alpha f(ax_1 + b) + (1-\alpha) f(ax_2 + b)
\end{aligned}
$$

(by the concavity of f)

$$= \alpha g(x_1) + (1-\alpha)g(x_2).$$

12. Determine the concavity/convexity of $f(x) = -(1/3)x^2 + 8x - 3$.

Solution

The function is twice-differentiable, because it is a polynomial. We have $f'(x) = -2x/3 + 8$ and $f''(x) = -2/3 < 0$ for all x, so f is strictly concave.

13. Let $f(x) = Ax^{\alpha}$, where $A > 0$ and α are parameters. For what values of α is f (which is twice differentiable) nondecreasing and concave on the interval $[0, \infty)$?

Solution

We have $f'(x) = \alpha Ax^{\alpha-1}$ and $f''(x) = \alpha(\alpha - 1)Ax^{\alpha-2}$. For any value of β we have $x^{\beta} \geq 0$ for all $x \geq 0$, so for f to be non decreasing and concave we need $\alpha \geq 0$ and $\alpha(\alpha - 1) \leq 0$, or equivalently $0 \leq \alpha \leq 1$.

14. Find numbers a and b such that the graph of the function $f(x) = ax^3 + bx^2$ passes through $(-1, 1)$ and has an inflection point at $x = 1/2$.

Solution

For the graph of the function to pass through $(-1,1)$ we need $f(-1) = 1$, which implies that $-a + b = 1$. Now, we have $f'(x) = 3ax^2 + 2bx$ and $f''(x) = 6ax + 2b$, so for f to have an inflection point at 1/2 we need

$f''(1/2) = 0$, which yields $3a + 2b = 0$. Solving these two equations in a and b yields $a = -2/5$, $b = 3/5$.

CHAPTER FOUR

4.1 CONCLUSION

Convex functions as we have established in the entire work can be said to be a very important tool in mathematics. A lot of facts have been established as a result of our study. We have seen that the intersection of two convex sets is convex and any affine set is convex but some convex sets are not affine. The vertices of any simplex are extreme points of the simplex.

Functions can be said to be either convex or concave. The negation of a convex function gives rise to a concave function. For a twice-differentiable function f, if the second derivative, $f''(x)$, is positive (or, if the acceleration is positive), then the graph is convex; if $f''(x)$ is negative, then the graph is concave. Points where concavity changes are inflection points. If a convex (i.e., concave upward) function has a "bottom", any point at the bottom is a minimal extremum. If a concave (i.e., concave downward) function has an "apex", any point at the apex is a maximal extremum.

If f(x) is a function of one variable and is single-peaked, then f(x) is quasi-concave. The sum of two convex functions is a convex and the sum of two concave functions is concave. A linear function is both convex and concave and a function can be neither convex nor concave. Functions of many variables can be written in a form of a Hessian matrix and from there we can tell if the function is convex or concave.

Convex and concave functions can be applicable in many sectors of life. From our work so far, we can see that convex and concave functions can be applied in solving problems from management, economics and in fact our everyday life. In analyzing graphs, the idea of convexity is used. Calculus also makes use of convex functions. Some of the most important applications of calculus require the use of the derivative of finding the maxima and minima. If we have functions that model cost, revenue, or population growth, for example, we can apply the methods of calculus to find the minima and maxima of the function. We have realized that a function when differentiated twice will give you a minima or a maxima.

4.2 RECOMMENDATIONS

For further studies, convex and concave functions should be well known to students. This makes the study of these functions relevant to all who aspire to apply them both theoretically and practically in areas like the industries, and many more.

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