Dama International Journal of Researchers (DIJR), ISSN: 2343-6743, ISI Impact Factor: 1.018
Vol 3, Issue 05, May, 2018, Pages 15 -29 , Available @ www.damaacademia.com

The Role of Concave and Convex Functions in the Study of Linear & Non-Linear Programming

Andrew Owusu-Hemeng[1,] Peter Kwasi Sarpong[2], Joseph Ackora-Prah[3]
[1,2,3]Department of Mathematics, Kwame Nkrumah University of Science and Technology,Kumasi,Ghana
Email: kp.sarp@yahoo.co.uk, owusuhemengandrew@gmail.com, ackph@yahoo.co.uk

I. INTRODUCTION

Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. It is very important in the study of optimization. The solutions to these problems lie on their vertices. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test. Miraculously, this has a natural generalization for the several variables case, the Hessian test. This project is intended to study the basic properties, some definitions, proofs of theorems and some examples of convex functions. Some definitions like convex and concave sets, affine sets, conical sets, concave functions shall be known. It will also prove that the negation of a convex function will generate a concave function and a concave set is a convex set. There is also the preposition that the intersection of convex sets is a convex set but the union of convex sets is not necessarily a convex set. This work is intended to help students acquire more knowledge on convex and concave functions of single variables. This will be done by differentiating the given function twice. If the second differential of the function is positive then we have a convex function. On the other hand if the second differential is negative then that function will be considered as concave. Examples will be solved to elaborate more on this. The convexity of functions of several variables will also be determined. This will be done by the use of the Hessian matrix. This will generate the idea of principal minors and leading principal minors. Firms can also use the idea of convex functions to know how they are doing in the market. Equations can be generated and with the help of curve sketching they will know if they are maximizing profits or making losses.

II. CONVEX SETS

A convex set is a set of elements from a vector space such that all the points on the straight line between any two points of the set are also contained in the set. If a and b are points in a vector space the points on the straight line between a and b are given by

$$x = \lambda a + (1 - \lambda) b \text{ for all } \lambda \text{ from 0 to 1}$$

The above definition can be restated as: A set S is convex if for any two points a and b belonging to S there are no points on the line between a and b that are not members of S.

Another restatement of the definition is: A set S is convex if there are no points a and b in S such that there is a point on the line between a and b that does not belong to S. The point of this restatement is to include the empty set within the definition of convexity. The definition also includes singleton sets where a and b have to be the same point and thus the line between a and b is the same point (Rawlins G.J.E. and Wood D, 1988).

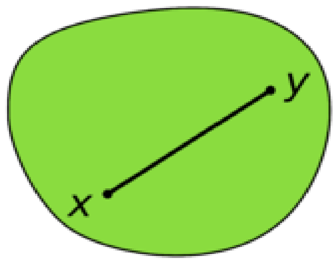


Figure 1 A convex set.

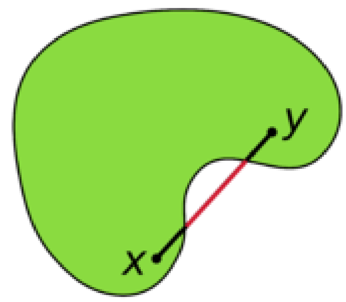


Figure 2 A non convex set

In figure 1, the line segment joining X and Y is contained in the set therefore it is a convex set. But in figure 2, part of the line segment is outside the set hence it's not a convex set. For example, the two-dimensional set on the left of the following figure is convex, because the line segment joining every pair of points in the set lies entirely in the set. The set on the right is not convex, because the line segment joining the points x and x' does not lie entirely in the set.

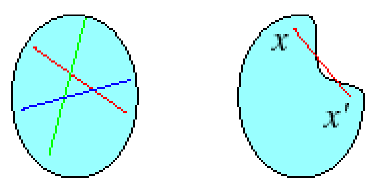


Figure 3 A convex and a non convex set

In Euclidean space, an object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object. For example, a solid cube is convex, but anything that is hollow or has a dent in it, for example, a crescent shape, is not convex.

A subset C of a Euclidean space is convex if it contains the line segment connecting any two of its members. That is, if x and y are vectors in C and t is a number between 0 and 1, the vector $tx + (1- t)\, y$ is also in C. A linear combination with non-negative weights which sum to 1 is convex combination of elements of C; a set C is convex if it contains all convex combinations of its elements (Munkres, James, 1999).

Let C be a set in a real or complex vector space. C is said to be convex if, for all x and y in C and all t in the interval [0,1], the point $(1 - t)\, x + t\, y$ is in C. In other words, every point on the line segment connecting x and y is in C. This implies that a convex set is connected.

The convex subsets of R (the set of real numbers) are simply the intervals of R. Some examples of convex subsets of Euclidean 2-space are regular polygons and bodies of constant width. Some examples of convex subsets of Euclidean 3-space are the Archimedean solids and the Platonic solids (Rawlins G.J.E. and Wood D).

A. Theorem

The intersection of any two convex sets is a convex set (Soltan, Valeriu, 1984). With the inclusion of the empty set as a convex set then it is true that:

The proof of this theorem is by contradiction. Suppose for convex sets S and T there are elements a *and* b such that a and b both belong to $S \cap T$, i.e., a belongs to S and T and b belongs to S *and* T and there is a point c on the straight line between a and b that does not belong to $S \cap T$. This would mean that c does not belong to one of the sets S *or* T

or both. For whichever set c does not belong to this is a contradiction of that set's convexity, contrary to assumption. Thus no such c and a and b can exist and $hence$ $S \cap T$ is convex.

In simple terms, the definition of convex sets can be summarized as follows;

- A set S is convex if the line segment between any two points in S lies in S.
- A set S is convex if every point in the set can be seen by every other point, along an unobstructed means lying in the set.
- A set S is said to be convex if whenever it contains two points a and b, it also contains the line segment joining them (Soltan, Valeriu, 1984).

B. Affine Sets

A set S is an affine set if and only if $x, y \in S$ and $\alpha, \beta \in R$,where $\alpha+\beta=1$, $\alpha x+\beta y \in S$. Thus, S is affine if and only if the line passing through any two distinct points of S is itself a subset of S (Moon, T., 2008).

Examples of affine sets are planes, lines, single points

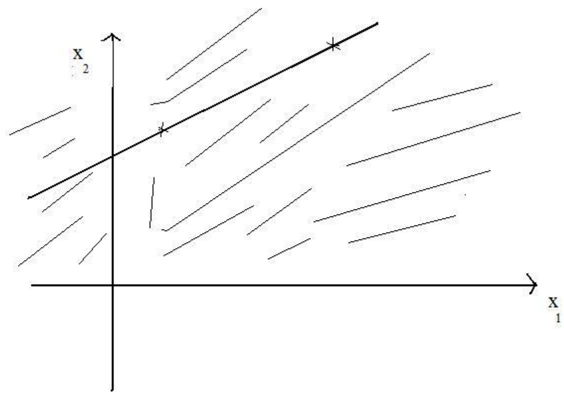
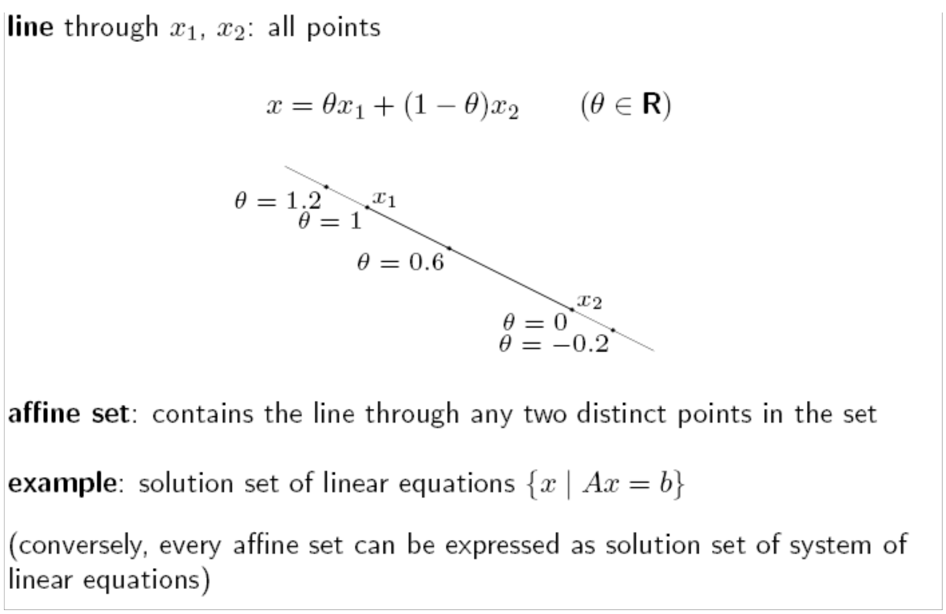


Figure 4 A non affine set

The above figure is not affine because the line passing through the two points extends outside the region.



C. Combinations

I. A linear combination of the vectors $\alpha_i \in R$.is any expression of the form $\sum \alpha_i x_i$ where $\alpha_i \in R$.

II. Affine combination of the vectors $x_{1,}\, x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\sum \alpha_i = 1,\ \ \alpha_i \in R$.

III. Conical combination of the vectors $x_{1,}\, x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\alpha_i \geq 0$, $\alpha_i \in R$..

IV. Convex combinations of the vectors $x_{1,}\, x_2, \dots, x_k$ is any expression of the form $\sum \alpha_i x_i$, where $\sum \alpha_i = 1$, $\alpha_i \geq 0$, $\alpha_i \in R$.

D. Hulls

I. Linear hull $[L(S)]$, is the collection of all linear combinations of the vectors of S.
i.e L(S) ={$\sum \alpha_i x_i$, each $x_i \in S$,each $\alpha_i \in R$.}

II. Affine hull $[Aff(S)]$, is the collection of all affine combinations of the vectors of S i.e Aff(S)={ $\sum \alpha_i x_i$, $each\ x_i \in S, each\ \alpha_i \in R, \sum \alpha_i = 1$.}

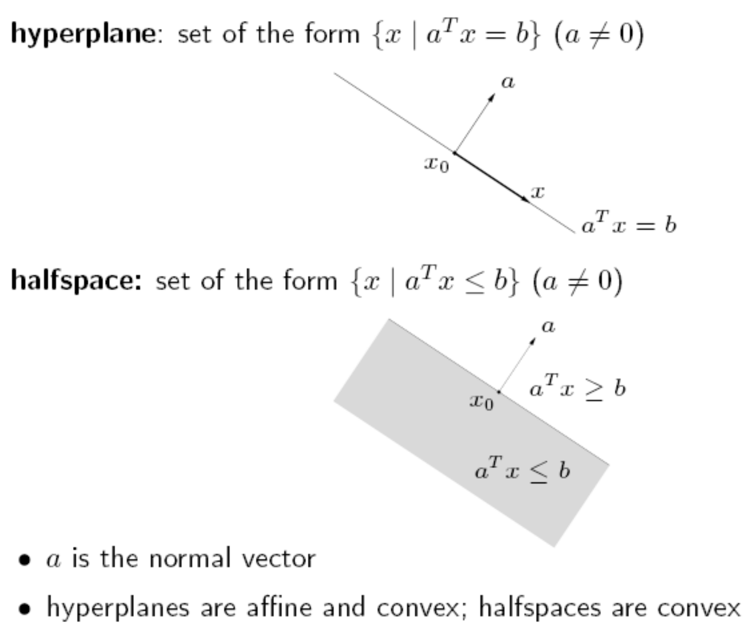
III. Conical hull [Coni(S)], is the collection of all conical combinations of the vectors of i.e Coni(S)={ $\sum \alpha_i x_i\ each\ x_i \in S, each\ \alpha_i \in R, \alpha_i \geq\ 0$.}

IV. Convex hull [Conv(S)], is the collection of all convex combinations of the vectors of S i.e Conv(S= {$\sum \alpha_i x_i, each\ x_i \in S, each\ \alpha_i \in R, \alpha_i \geq 0,\ \sum \alpha_i = 1$}

III. HYPERPLANE AND HALFSPACES

Let C denote a non zero vector in R^n,then the set $\{x \in R^n : c^T x = k\}$ is a Hyperplane in R. Thus in R^2,a hyper plane is the set of all points (x_1, x_2) such that $c_1x_1+c_2x_2=k$ where $(c_1,c_2)^T \neq 0$,that is, a hyperplane in R^2 is a straight line. In R^3, take c= $(c_1,c_2,c_3)^T$ then a hyperplane in R^3 is the set of all points $(x_1,x_2,x_3)^T$ such that $c_1x_1+c_2x_2+c_3x_3=k$. Thus a hyperplane in R^3 is a plane (Bertsekas, Dimitri, 2003).

Let H=$\{x \in R^n : c^T x = k\}$ denote a hyperplane in R^n. If R^n is divided by H into two parts$\{x \in R^n : c^T x \leq k\}$ and $\{x \in R^n : c^T x \geq k\}$. Each of these parts is called a closed half space. R^n can also be divided into three disjoint parts by the hyperplane H namely H itself $\{x \in R^n : c^T x = k\}$, $\{x \in R^n : c^T x < k\}$ and $\{x \in R^n : c^T x > k\}$. Each of these last two parts is called an open half (Bertsekas, Dimitri, 2003).



A. Cones

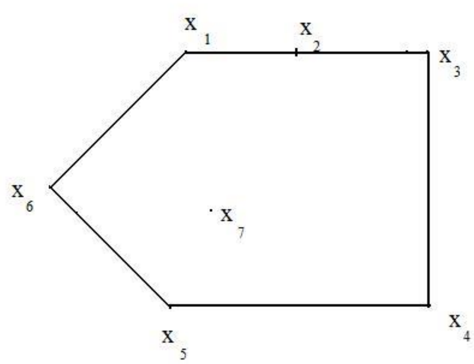
A set K is a cone if $\lambda x \in K$ for each $x \in K$ and $\lambda \geq 0$. A convex cone is a cone which is a convex set. A set K is a convex cone if it is convex and a cone, which means that for any $x_1, x_2 \in K$ and $\lambda_1, \lambda_2 \geq 0$ we have$\lambda_1 x_1 + \lambda_2 x_2 \in K$.

A point of the form $\lambda_1 x_1 + \cdots + \lambda_n x_n$ with $\lambda_1, \ldots, \lambda_k \geq 0$ or $[\sum \lambda_i x_i, where \lambda_i \geq 0]$ is called a conic combination of $x_1, \ldots, x_n$. If x_i are in a convex cone K, then every conic combination of x_i is in K. Conversely, a set K is a convex cone if and only if, it contains all conic combinations of its elements.
A finite cone is the sum of a finite number of rays.
1. The intersection of an arbitrary collection of convex cones is a convex cone.
2. A finite cone is a closed convex cone.
Geometrically, a convex cone with vertex the origin is any convex set that contains all the rays from the origin through each of the points in the set. The set

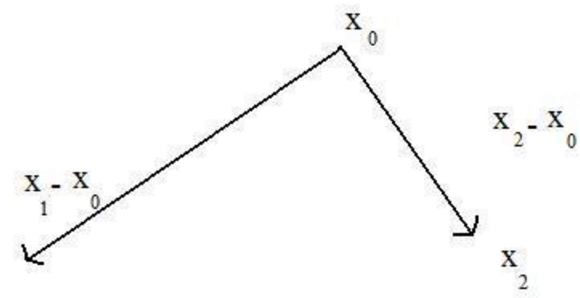
$S = \{(x_1, x_2)^T : x_1 = |x_2|\}$ is not a convex cone with vertex the origin since it is not a convex set. The set $T = \{(x_1, x_2)^T : x_1 \geq |x_2|$ is such a cone (Hiriart-Urruty *et al*, 2004).

B. Convex Polytope

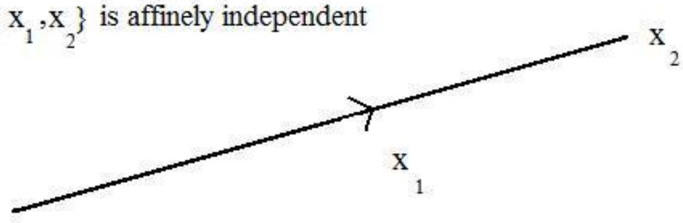
Convex set play a very important role in mathematical programming. The convex hull of finitely many points is called a convex polytope. The set of vectors $\{x_0, x_1, \ldots, x_k\}$ is affinely independent if and only if the set$\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent (Hiriart-Urruty *et al*, 2004). A convex polytope is called a k-dimensional simplex if and only if it is of $k + 1$ affinely independent vectors. A convex polytope is an enclosed figure. The diagram below is an example of a convex polytope:



$$S = conv\{x_1, x_2, \ldots, x_7\}$$



$\{x_0, x_1, x_2\}$ is affinely independent

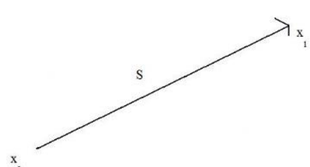


$\{x_0, x_1, x_2\}$ is not affinely independent

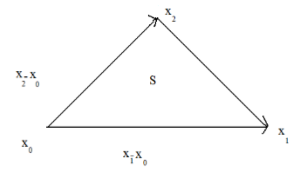
C. Simplex

A simplex is an n-dimensional convex polytope having exactly $(n + 1)$ vertices.
A set in R^n which is a convex hull of $(n + 1)$ points in R^n is called n-simplex.
For instance;
A simplex in 0-dimension is a point.

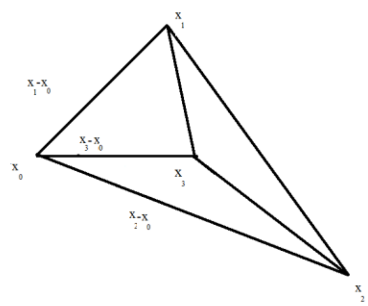
a simplex in 1-dimension is a line.



a simplex in 2-dimension is a triangle

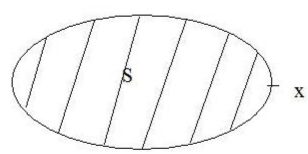


a simplex in 3-dimension is a tetrahedron

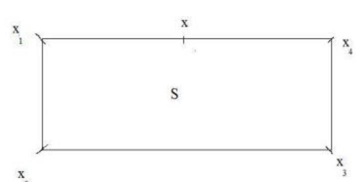


D. Extreme Points

The extreme points of a figure are the points on the boundary of that figure. Examples can be seen in the figures below:



The extreme point of this figure is the point x lying on its boundary.



The extreme points of S are x_1, x_2, x_3, x_4 and x on the boundary.
Clearly, the extreme points of a convex set S are those points of S that do not lie on the interior of any line segment connecting any other pair of points of S(Luenberg D., 1984).

E. Examples of Convex Sets

1. R^n
2. Point
3. Hyperplane or closed or open half space

4. Line

5. Line segment

V. CONCAVE SETS

In mathematics, the notion of concave set is complementary to that of the convex set.
The following definitions are in use.

- A set is called concave if it is not convex.
- A set is called concave if its complement is convex.

A. Convex Function

In mathematics, a real-valued function f defined on an interval (or on any convex subset of some vector space) is called convex, concave upwards, concave up or convex cup. If for any two points x and y in its domain C and any t in [0,1], we have

$$f\ (tx + (1-t)y)\ \leq\ tf(x)+(1-t)f(y)$$

In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set. Pictorially, a function is called 'convex' if the function lies below the straight line segment connecting two points, for any two points in the interval [0,1].

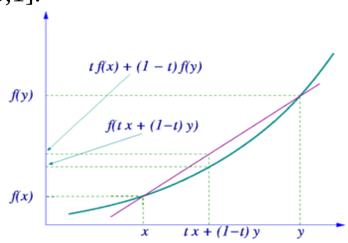


Figure 5 A convex function

A function is called strictly convex if

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

for any $t\ in\ (0,1)\ and\ x \neq y$.
Let f be a function of several variables, defined on a convex set S. We say that f is convex if the line segment joining any two points on the graph of f is never below the graph.

Definition

Let f be a function of several variables defined on the convex set S. Then f is

- convex on the set S if for all $x \in S$, all $x' \in S$, and all $\lambda \in (0,1)$ we have
  $f\ ((1-\lambda)x\ +\ \lambda x')\ \leq\ (1-\lambda)\ f\ (x)\ +\ \lambda\ f\ (x')$.
- strictly convex function is one that satisfies the definition for concavity with a strict inequality (< rather than ≤) for all $x\ \neq x'$.

Definition

- If a convex (i.e., concave upward) function has a "bottom", any point at the bottom is a minimal extremum. If a concave (i.e., concave downward) function has an "apex", any point at the apex is a maximal extremum.
- If $f(x)$ is twice-differentiable, then $f(x)$ is concave if and only if $f\,''(x)$ is non-positive. If its second derivative is negative then it is strictly concave, but the opposite is not true, as shown by $f(x)\ =\ -x^4$.

Examples of Convex Functions

- The function $f(x) = x^2\ has\ f''(x) = 2 > 0$ at all points, so f is a (strictly) convex function.
- The absolute value function $f(x) = |x|$ is convex, even though it does not have a derivative at the point $x = 0$.
- The function $f(x) = |x|\ p\ for\ 1 \leq p$ is convex.
- The exponential function $f(x) = e^x$ is convex. More generally, the function $g(x) = e^{f(x)}$ is logarithmically convex if f is a convex function.
- The function f with domain [0,1] defined by $f(0) = f(1) = 1, f(x) = 0$ for $0 < x < 1$ is convex; it is continuous on the open interval (0,1), but not continuous at $0\ and\ 1$.
- The function x^3 has second derivative $6x$; thus it is convex on the set where $x \geq 0$ and concave on the set where $x \leq 0$.
- Every norm is a convex function, by the triangle inequality.

B. Concave Functions

In mathematics, a concave function is the negative of a convex function. A concave function is also synonymously called concave downwards, concave down or convex cap (Rockafellar, R. T.,1970).

Formally, a real-valued function f defined on an interval (or on any convex set C of some vector space) is called concave, if for any two points $x\ and\ y$ in its domain C and any t in [0,1], we have

$$f\ (tx + (1-t)y) \ \geq\ tf(x) + (1-t)f(y)$$

Also, $f(x)$ is concave on $[a, b]$ if and only if the function $-f(x)$ is convex on $[a, b]$.

A function is called strictly concave if

$$f\ (tx + (1-t)y) > tf(x) + (1-t)f(y)$$

for any $t\ in\ (0,1)\ and\ x \neq y$.

This definition according to Rockafellar, R. T. (1970), merely states that for every z between $x\ and\ y$, the point $(z, f(z)\ )$ on the graph of f is above the straight line joining the points $(x, f(x)\ )$ and $(y, f(y)\ )$.

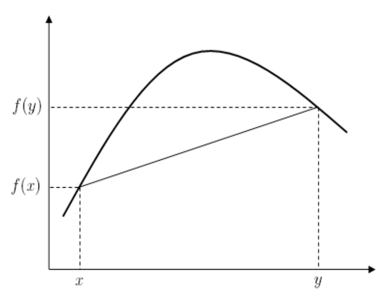


Figure 6 A concave function

A continuous function on C is concave if and only if

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

for any $x\ and\ y\ in\ C$.

A differentiable function f is concave on an interval if its derivative function f ′ is monotonically decreasing on that interval: a concave function has a decreasing slope. ("Decreasing" here means "non-increasing", rather than "strictly decreasing", and thus allows zero slopes.)

Let f be a function of many variables, defined on a convex set *S*. We say that f is concave if the line segment joining any two points on the graph of f is never above the graph.
A function f is said to be concave if $-f$ is convex (Rawlins G.J.E. and Wood D,1988).

Definition
Let f be a function of many variables defined on the convex set *S*. Then f is

- concave on the set *S* if for all $x \in S, all\ x' \in S$, and all $\lambda \in (0,1)$ we have
  $f\,((1-\lambda)x + \lambda x') \geq (1-\lambda)\, f\,(x) + \lambda\, f\,(x')$.

Once again, a strictly concave function is one that satisfies the definition for concavity with a strict inequality (> rather than ≥) for all $x \neq x'$.

Proposition
A function f of many variables defined on the convex set *S* is

- concave if and only if the set of points below its graph is convex:
  $\{(x, y): x \in S\ and\ y \leq f\,(x)\}$ is convex

Properties
For a twice-differentiable function f, if the second derivative, $f\,''(x)$, is positive (or, if the acceleration is positive), then the graph is convex; if $f\,''(x)$ is negative, then the graph is concave. Points where concavity changes are inflection points. (Stephen Boyd and Lieven Vandenberghe 2004).

Quasi-Concave
A function $f(x)$ is quasi-concave if $f(x) \geq f(y)\ implies\ f(\,tx + (1-t)y\,) \geq f(y)$

- If $f(x)$ is a function of one variable and is single-peaked, then $f(x)$ is quasi-concave.
- If $f(x)$ is quasi-concave, then its upper level sets are convex sets. The level curves (isoquants, indifference curves) are convex to the origin (diminishing marginal rate of substitution).
- If $f(x)$ is quasi-concave, then the Hessian matrix is negative semi-definite subject to constraints. In particular, let H be the Hessian matrix of $f(x)$, and let f' be the vector of first derivatives of $f(x)$, then

$$x'\,H\,(x) \leq 0\ for\ all\ f'\,(x) = 0$$

If $f(x)$ is quasi-concave, then the bordered Hessian matrix

[f_11 f_12 ... f_1n f_1]
[f_21 f_22 ... f_2n f_2]

H = [..........................]
[f_n1 f_n2 ... f_nn f_n]
[f_1   f_2   ... f_n   0   ]

have border-preserving principle minors determinants of order k that alternate in sign. i.e., the $3x3$ (including the border) determinant is positive, the 4 x 4 determinant is negative, and so on. Note that quasi-concavity does not imply that $f_{11} \leq 0$. If $f(x)$ is quasi-concave, then $-f(x)$ is quasi-convex. The lower level sets of a quasi-convex function are convex (Press, W. H *et al.*, 1995).

Relationship between concavity and quasi-concavity:

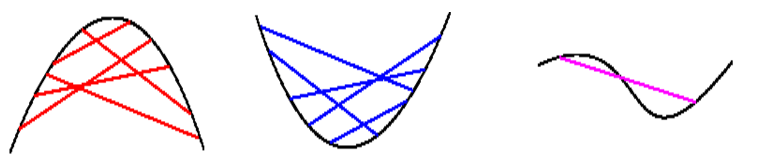
- All concave functions are quasi-concave.

- Any monotonic transformation of concave function is quasi-concave.
- Quasi-concave functions are not necessarily concave.

A set S is convex if for any two elements x *and* y that belongs to S, the element $tx + (1-t)\,y$ also belongs to S (t *is between* 0 *and* 1).

C. Concave and Convex Functions Of Single Variables

The twin notions of concavity and convexity are used widely in economic theory, and are also central to optimization theory. A function of a single variable is concave if every line segment joining two points on its graph does not lie above the graph at any point. Symmetrically, a function of a single variable is convex if every line segment joining two points on its graph does not lie below the graph at any point. These concepts are illustrated in the following figure.



A concave function.
No line segment lies above the graph at any point.

A convex function.
No line segment lies below the graph at any point.

A function that is neither concave nor convex.
The line segment shown lies above the graph at some points and below it at other points.

Figure 7 Function of a single variable

Here is a precise definition.

Note that a function may be both concave and convex. Let f be such a function. Then for all values of a and b we have

$$f\,((1-\lambda)a + \lambda b) \geq (1-\lambda)\,f\,(a) + \lambda\,f\,(b)\;for\;all\;\lambda \in (0,1)\;and$$
$$f\,((1-\lambda)a + \lambda b) \leq (1-\lambda)\,f\,(a) + \lambda\,f\,(b)\;for\;all\;\lambda \in (0,1).$$

Equivalently, for all values of *a* and *b* we have

$f\,((1-\lambda)a + \lambda b) = (1-\lambda)\,f\,(a) + \lambda\,f\,(b)$ for all $\lambda \in (0, 1)$. That is, a function is both concave and convex if and only if it is linear, taking the form $f(x) = ax + b$ for all *x*, for some constants a and b.

Proof

$$f(\alpha x' + (1-\alpha)x") = a[\alpha x' + (1-\alpha)x"] + b$$
$$= [ax' + b] + (1-\alpha)(ax" + b)$$
$$= \alpha f(x') + (1-\alpha)f(x")$$

Since both definitions of convex and concave functions are satisfied with equality, *f(x) = ax+b* is both a convex and a concave function.

D. Convexity Of Functions Of Several Variables

To determine whether a twice-differentiable function of many variables is concave or convex, we need to examine all its second partial derivatives. We call the matrix of all the second partial derivatives the Hessian of the function.

Let f be a twice differentiable function of n variables. The Hessian of f at x is

$$H(x) = \begin{bmatrix} f_{11}"(x) & f_{12}"(x) & \ldots & f_{1n}"(x) \\ f_{21}"(x) & f_{22}"(x) & \ldots & f_{2n}"(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_{n1}"(x) & f_{n2}"(x) & \ldots & f_{nn}"(x) \end{bmatrix}$$

1. The hessian of $f(x_{1,}\,x_2, \ldots x_n)$ is the $n \times n$ matrix whose ij^{th} entry is $\frac{\partial^2}{\partial x_i \partial y_i}$

Example

Let $H(x_{1,}x_2, \ldots x_n)$ denote the value of the Hessian at (x_1, x_2) if $f(x_{1,}x_2)= x_1^3 + 2x_1x_2 + x_2^2$ then $H(x_1, x_2) = \begin{bmatrix} 6x_1 & 2 \\ 2 & 2 \end{bmatrix}$

2. An i^{th} principal minor of an $n \times n$ matrix is the determinant of any $i \times i$ matrix obtained by deleting $n-i$ rows and the corresponding$(n-i)$ columns of the matrix.

   Example

   The matrix$\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$ has first principal minors as -2 and -4 and the 2^{nd} principal minor is $-2(-4) - (-1)(-1) = 7$.

3. The K^{th} leading principal minor of an $n \times n$ matrix is the determinant of the $K \times K$ matrix obtained by deleting the last $(n-k)$ rows and columns of the matrix.

   Example

   Let $H_k(x_{1,}x_2, \ldots, x_n)$ be K^{th} leading principal minor of the Hessian matrix evaluated at the point $(x_1, x_2, \ldots, x_n)$. Then if $f(x_1, x_2) = x_1^3 + 2x_1x_2 + x_2^2$ then $H_1(x_1, x_2) = 6x_1$
   $H_2(x_1, x_2) = 6x_1(2) - 2(2) = 12x_1 - 4$.

E. Theorem

Suppose $f(x_1, x_2, \ldots, x_n)$ has continuous second-order partial derivatives for each point $(x_1, x_2, \ldots, x_n) \in S$. then $f(x_1, x_2, \ldots, x_n)$ is a convex function S if and onl if for each $x \in S$, all principal minors of$H$ are non−negative.

Suppose $f(x_1, x_2, \ldots, x_n)$ has continuous second-order partial derivatives for each point $x = (x_1, x_2, \ldots, x_n) \in S$. Then $f(x_1, x_2, \ldots, x_n)$ is a concave function on S if and only if for each $x \in S \text{ and } K = 1,2, \ldots, n$, all non-zero principal minors have the same sign as$(-1)^K$.

B applying theorem (1) and (2) above, the Hessian matrix can be used to determine whether $f(x_1, x_2, \ldots, x_n)$a convex is or a concave function on a convex set $S\subset R^n$.

Examples

I. Show that $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ is a convex function on $S = R^2$.

Solution

$H(x_1, x_2) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

1st principal minors of the Hessian are the diagonal entries and both are ≥0. The 2nd principal minor is $2(2) - 2(2) = 0 \geq 0$. Since for any point, all principal minors of H are non-negative, theorem (1) shows that $f(x_1, x_2) = -x_1^2 - x_1x_2 - 2x_2^2$

Is a concave function on R^2.

II. Show that for $S = R^2$, $f(x_1,x_2) = x_1^2 - 3x_2x_2 + 2x_2^2$ is not a convex or a concave function.

Solution

$H(x_1,x_2) = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

1st principal minors of the Hessian are the diagonal entries and both of them ≤ 0. The 2nd principal minor is

$2(2) - (-3)(-3) = 5 \geq 0$. since the 1st principal minor is negative and the 2nd principal minor is

positive, the function is neither convex nor concave.

Consider the function $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3$ defined on the set of all triples of numbers. Its first partials are

$f'_1(x_1, x_2, x_3) = 2x_1 + 2x_2 + 2x_3$
$f'_2(x_1, x_2, x_3) = 4x_2 + 2x_1$
$f'_3(x_1, x_2, x_3) = 6x_3 + 2x_1$.

So its Hessian is

$$\begin{pmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{pmatrix} = Z \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

The leading principal minors of the Hessian are $2 > 0, 4 > 0, and\ 6 > 0$. So the Hessian is positive

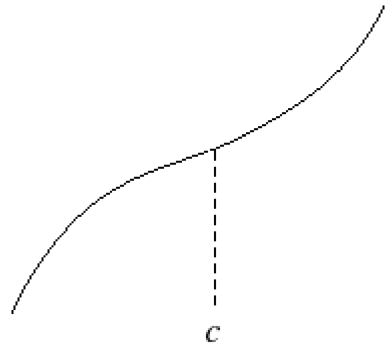
definite, and f is strictly convex.

Definition

A point at which a twice-differentiable function changes from being convex to concave, or vice versa, is an inflection point. c is an inflection point of a twice-differentiable function f of a single variable if for some values of a and b with $a < c < b$ we have

- either $f''(x) \geq 0$ if $a < x < c$ and $f''(x) \leq 0$ if $c < x < b$
- or $f''(x) \leq 0$ if $a < x < c$ and $f''(x) \geq 0$ if $c < x < b$.

An example of an inflection point is shown in the following figure.



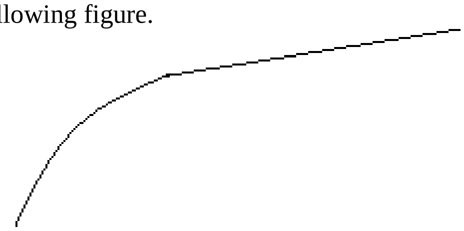
Proposition

- If c is an inflection point of f then $f''(c) = 0$.
- If $f''(c) = 0$ and f'' changes sign at c then c is an inflection point of f.

Note, however, that f'' does not have to change sign at c for c to be an inflection point of f. For example, every point is an inflection point of a linear function.

F. Strict Convexity and Concavity

The inequalities in the definition of concave and convex functions are weak: such functions may have linear parts, as in the following figure.



a concave, but not strictly concave, function

A concave function that has no linear parts is said to be strictly concave.

Definition

The function f of a single variable defined on the interval I is

- strictly concave if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have $f((1-\lambda)a + \lambda b) > (1 - \lambda) f(a) + \lambda f(b)$.
- strictly convex if for all $a \in I$, all $b \in I$ with $a \neq b$, and all $\lambda \in (0,1)$ we have $f((1-\lambda)a + \lambda b) < (1 - \lambda) f(a) + \lambda f(b)$.

An earlier result states that if f is twice differentiable then f is concave on $[a, b]$ if and only if $f''(x) \leq 0$ for all $x \in (a, b)$. Does this result have an analogue for *strictly* concave functions? Not exactly. If $f''(x) < 0$ for all $x \in (a,b)$ then f is strictly concave on $[a, b]$, but the converse is not true: if f is strictly concave then its second derivative is not necessarily negative at all points. (Consider the function $f(x) = -x^4$ It is concave, but its second derivative at 0 is zero, not negative) That is, f is strictly concave on $[a, b]$ if $f''(x) < 0$ for all $x \in (a, b)$, but if f is strictly concave on $[a, b]$ then $f''(x)$ is not necessarily negative for all $x \in (a, b)$. (Analogous observations apply to the case of convex and strictly convex functions, with the conditions $f''(x) \geq 0$ and $f''(x) > 0$ replacing the conditions $f''(x) \leq 0$ and $f''(x) < 0$)

VI. CONCLUSION

Convex functions as we have established in the entire work can be said to be a very important tool in mathematics. A lot of facts have been established as a result of our study. We have seen that the intersection of two convex sets is convex and any affine set is convex but some convex sets are not affine. The vertices of any simplex are extreme points of the simplex.

Functions can be said to be either convex or concave. The negation of a convex function gives rise to a concave function. For a twice-differentiable function f, if the second derivative, $f''(x)$, is positive (or, if the acceleration is positive), then the graph is convex; if $f''(x)$ is negative, then the graph is concave. Points where concavity changes are inflection points. If a convex (i.e., concave upward) function has a "bottom", any point at the bottom is a minimal extremum. If a concave (i.e., concave downward) function has an "apex", any point at the apex is a maximal extremum. If f(x) is a function of one variable and is single-peaked, then f(x) is quasi-concave. The sum of two convex functions is a convex and the sum of two concave functions is concave. A linear function is both convex and concave and a function can be neither convex nor concave. Functions of many variables can be written in a form of a Hessian matrix and from there we can tell if the function is convex or concave.

Convex and concave functions can be applicable in many sectors of life. From our work so far, we can see that convex and concave functions can be applied in solving problems from management, economics and in fact our everyday life. In analyzing graphs, the idea of convexity is used. Calculus also makes use of convex functions. Some of the most important applications of calculus require the use of the derivative of finding the maxima and minima. If we have functions that model cost, revenue, or population growth, for example, we can apply the methods of calculus to find the minima and maxima of the function. We have realized that a function when differentiated twice will give you a minima or a maxima.

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