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Applications of Convex Function and Concave Functions

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Abstract
In recent years, convex optimization has become a computational tool of central importance in mathematics and economics, thanks to its ability to solve very large, practical mathematical problems reliably and efficiently. The goal of this project is to give an overview of the basic concepts of convex sets, functions and convex optimization problems, so that the reader can more readily recognize and formulate basic problems using modern convex optimization. This helps in solving real world problems.

I. INTRODUCTION

Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. It is very important in the study of optimization. The solutions to these problems lie on their vertices.

The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test. Miraculously, this has a natural generalization for the several variables case, the Hessian test.

This project is intended to study the basic properties, some definitions, proofs of theorems and some examples of convex functions. Some definitions like convex and concave sets, affine sets, conical sets, concave functions shall be known. It will also prove that the negation of a convex function will generate a concave function and a concave set is a convex set. There is also the preposition that the intersection of convex sets is a convex set but the union of convex sets is not necessarily a convex set.

This work is intended to help students acquire more knowledge on convex and concave functions of single variables. This will be done by differentiating the given function twice. If the second differential of the function is positive then we have a convex function. On the other hand if the second differential is negative then that function will be considered as concave. Examples will be solved to elaborate more on this.

The convexity of functions of several variables will also be determined. This will be done by the use of the Hessian matrix. This will generate the idea of principal minors and leading principal minors.

Firms can also use the idea of convex functions to know how they are doing in the market. Equations can be generated and with the help of curve sketching they will know if they are maximizing profits or making losses.

II. METHODOLOGY

Convex function is itself a mathematical tool. Most of our findings will be from literature from the library. It will also be from information gathered from the internet and consultations from people who have ideas on convex functions.

III. APPLICATIONS OF CONVEX FUNCTION AND CONCAVE FUNCTIONS

As a matter of fact, we experience convexity all the time and in many ways. The most prosaic example is our upright position, which is secured as long as the vertical projection of our center of gravity lies inside the convex envelope of our feet. Also, convexity has a great impact on our everyday life through numerous applications in industry, business, medicine, and art. So do the problems of optimum allocation of resources and equilibrium of no-cooperative games. We shall limit ourselves to just a few of them.

Economists often assume that a firm's production function is increasing and concave. An example of such a function for a firm that uses a single input is shown in the next figure. The fact that such a production function is increasing means that more input generates more output. The fact that it is concave means that the increase in output generated by a one-unit increase in the input is smaller when output is large than when it is small. That is, there are "diminishing returns" to the input, or, given that the firm uses a single input, "diminishing returns to scale". For some (but not all) production processes, this property seems reasonable.

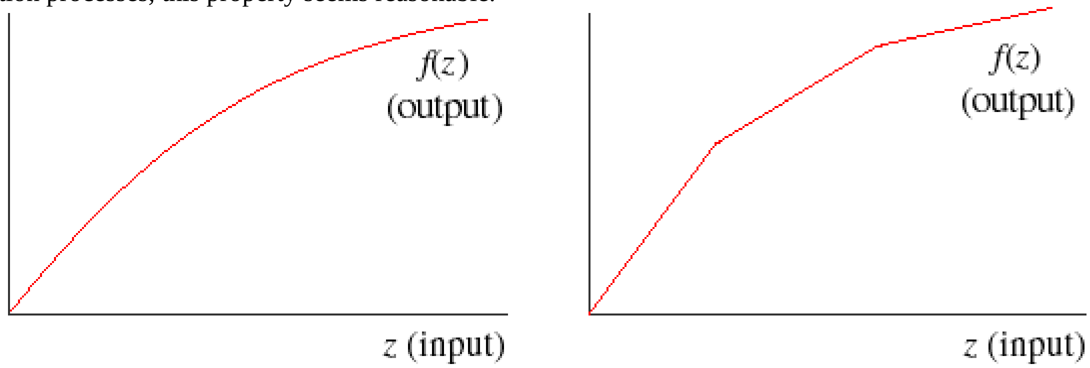


Figure 1 Two concave production functions

The notions of concavity and convexity are important in optimization theory because, as we shall see, the first-order conditions are sufficient (as well as necessary) for a maximizer of a concave function and for a minimizer of a convex function. (Precisely, every point at which the derivative of a concave differentiable function is zero is a maximizer of the function, and every point at which the derivative of a convex differentiable function is zero is a minimizer of the function.)

Three functions of importance to an economist or a manufacturer

$C(x)$ =total cost of producing x units of a product during some time period

$R(x)$ = total revenue received from selling x units of the product during the time period

$P(x)$ =total profit obtained by selling x units of the product during the time period

These are called, respectively, the cost function, revenue function, and profit function. If all units produced are sold, then these are related by

$$P(x) = R(x) - C(x)$$

That is profit = revenue−cost

The total cost $C(x)$ of producing x units can be expressed as a sum

$$C(x) = a + M(x)$$

Where **a** is a constant, called overhead, and $M(x)$ is a function representing manufacturing cost. The overhead,

which includes such fixed costs as rent and insurance, does not depend on x; it must be paid even if nothing is produced. On the other hand, the manufacturing cost M(x), which includes such items as cost of materials and labour, depends on the number of items manufactured. It is shown in economics that with suitable simplifying assumptions, $M(x)$ can be expressed in the form

$$M(x) = bx + cx^2$$

Where b and c are constants. Substituting this in the previous equation yields

$$C(x) = a + bx + cx^2$$

If a manufacturing firm can sell all the items it produces for p dollars apiece, then its total revenue *R(x)* (in dollars) will be

$R(x) = px$

And its total profit *P(x)* (in dollars) will be

$$P(x) = [total\ revenue] - [total\ cost] = R(x) - C(x) = px - C(x)$$

Thus, the cost function is given by

$$P(x) = px - (a + bx + cx^2)$$

Depending on such factors as number of employees, amount of machinery available, economic conditions, and competition, there will be some upper limit l on the number of items a manufacturer is capable of producing and selling. Thus, during a fixed time period the variable x in the previous equation will satisfy

$$0 \leq x \leq l$$

By determining the value or values of x in $[0,l]$ that maximize the previous equation, the firm can determine how many units of its product must be manufactured and sold to yield the greatest profit.
This is illustrated in the example below.

Examples

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is

$$C(x) = 500{,}000 + 80x + 0.003x^2$$

And if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit?

Solution.

Since the total revenue for selling x units is $R(x) = 200x$, the profit $P(x)$ on x units will be

$$P(x) = R(x) - C(x) = 200x - (500{,}000 + 80x + 0.003x^2)$$

Since the production capacity is at most 30,000 units, x must lie in the interval [0,30,000].

$$dP/dx = 200 - (80 + 0.006x) = 120 - 0.006x$$

$$setting\ dP/dx = 0\ gives\ 120 - 0.006x = 0\ \ or\ \ x = 20{,}000$$

since this critical point lies in the interval [0,30,000], the maximum profit must occur at one of the points

$x = 0$, $x = 20{,}000$, or $x = 30{,}000$

substituting these values yields the table below, which tells us that the maximum profit $P = \$700{,}000$ occurs when $x = 20{,}000$ units are manufactured and sold in the specified time.

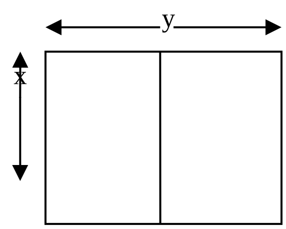
x	0	20,000	30,000

P(x)	−500,000	700,000	400,000

Question
A competitive firm receives the price $p > 0$ for each unit of its output, and pays the price $w > 0$ for each unit of its single input. Its output from using x units of the variable input is $f(x) = x^{1/4}$. Is this production function concave? Is the firm's profit concave in x?

Solution
The function f is twice-differentiable for $x > 0$. We have $f'(x) = (1/4)x^{-3/4}$ and $f''(x) = -(3/16)x^{-7/4} < 0$ for all x, so f is concave for $x > 0$. It is continuous, so it is concave for all $x \geq 0$. The firm's profit, $p f(x) - wx$, is thus the sum of two concave functions, and is hence concave. Often in life, we are faced with the problem of finding the best way to do something. For example, a farmer wants to choose the mix of crops that is likely to produce the largest profit. A doctor wishes to select the smallest dosage of drug that will cure a certain disease. A manufacturer would like to minimize the cost of distributing its products. Most at times problems of this nature can be formulated so it involves maximizing or minimizing a function over a specified set. Convex and Concave functions are best used for problems of this nature. Suppose then that we have a function f and a domain S. The first thing to do is to decide whether f has a maximum value or a minimum value on S. Secondly, assuming that such values exist, we are interested in knowing where on S they are attained. Finally, we wish to determine the maximum and the minimum values.

Question
Farmer Brown has 100 meters of wire fence with which he plans to build two identical adjacent pens, as shown in the diagram below. What are the dimensions of the total enclosure for which its area is a maximum?



Solution
let x be the width and y the length of the total enclosure, both in meters. Because there are 100 meters of fence, 3x+2y=100 that is y=50−3/2x. The total area, A is given by $A = xy = 50x - \frac{3}{2}x^2$. Since there must be three sides of length x, we see that $0 \leq x \leq \frac{100}{3}$. Thus, our problem is to maximize A on $[0, \frac{100}{3}]$. Now,

$\frac{dA}{dx} = 50 - 3x$ when we set $50 - 3x$ equal to 0 and solve, we get $x = \frac{50}{3}$ as a stationary point. Thus, there are three critical points $0, \frac{50}{3}$ and $\frac{100}{3}$. The two endpoints 0 and $\frac{100}{3}$ give A=0. While $x = \frac{50}{3}$ yields y=50−$\frac{3}{2}(\frac{50}{3}) = 25$meters.

Question
Convexity can be used to find local extreme values. For example find the local extreme values of the function $f(x) = x^2 - 6x + 5$ on(−∞,∞).

Solution;

The polynomial function f is continuous everywhere, and its derivative, $f'(x) = 2x - 6$, exists for all x. Thus, the only critical point for f is the single solution of $f'(x) = 0$, namely, $x = 3$. Since $f'(x) = 2(x-3) < 0 \; for \; x < 3$, f is decreasing on (−∞,3];and because $2(x-3) > 0$ for x>0, f is increasing on [3,∞). Therefore, by the first derivative test, $f(3) = -3$ is a local minimum value of f. Since 3 is the only critical number, there are no other extreme values. The graph of ***f*** is shown in the figure below. Note that f(3) is actually the global minimum value in this case.

Question
For $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$, use the second derivative test to identify local extrema.

Solution:

$$f'(x) = x^2 - 2x - 3 = (x+1)(x-3)$$

$$f''(x) = 2x - 2$$

The critical points are $-1 \; and \; 3(f'(-1) = f'(3) = 0)$.

$Since \; f''(-1) = -4 \; and \; f''(3) = 4$, we conclude by the second derivative test that $f(-1)$ is a local maximum value and that f(3) is a local minimum value.

Question
Finding extrema on open intervals Find(if possible) the minimum and maximum values of $f(x) = x^4 - 4x$ on (−∞,∞).

Solution:

$$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2 + x + 1)$$

since $x^2 + x + 1 = 0$ has no real solutions(quadratic formula), there is only one critical point, namely, $x = 1. \; For \; x < 1, f'(x) < 0$, whereas

for $x > 1, f'(x) > 0$. We conclude that $f(1) = -3$ is a local minimum value for f; and since f is decreasing on the left of 1 and increasing on the right of 1, it must actually be the minimum value of f. This implies that f cannot have a maximum value. The graph is shown below.

Question
Practical problem

A rectangular beam is to be cut from a log with circular cross section. If the strength of the beam is proportional to the product of its width and the square of its depth, find the dimensions of the cross section that give the strongest beam.

Solution

Denote the diameter of the log by a(constant) and the width and depth by w and d, respectively(as in the figure).

We want to maximize S, the strength of the beam. From the conditions given in the problem,

$$S = kwd^2$$

Where k is a constant of proportionality. The strength S depends on the two variables w and d, but there is a simple relationship between them.

$$d^2 + w^2 = a^2$$

when we solve this equation for d^2 and substitute in the formula for S in terms of the single variable w.

$$S = kw(a^2 - w^2) = ka^2w - kw^3$$

We consider the allowable values for w to be 0<w<a, an open interval. To find the critical points we calculate dS/dw , set it equal to 0, and solve for w.

$$dS/dw = ka^2 - 3kw^2 = k(a^2 - 3w^2)$$

$$k(a^2 - 3w^2) = 0$$

$$3w^2 = a^2$$

$$w^2 = a^2/3$$

$$w = a/\sqrt{3}$$

$Since\ a/\sqrt{3}$ is the only critical point in (0,a), it is likely that it gives the maximum S. when we substitute $w = a/\sqrt{3}\ in\ d^2 + w^2 = a^2$ we learn that d=√2a/√3. The desired dimensions are w= a/√3 and d=√2a/√3 and d=√2w.

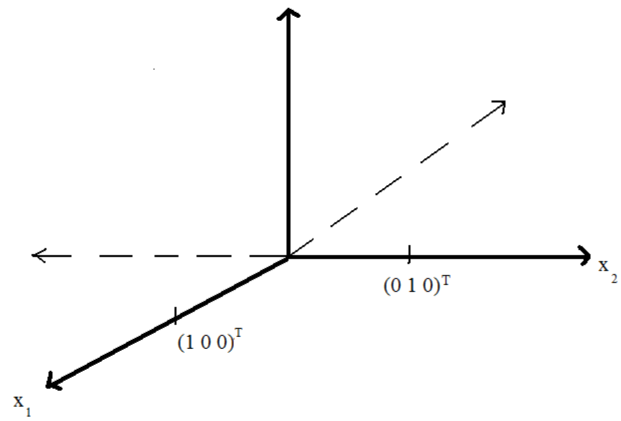
A. Exercises

1. Let S= $\{(1\ 0\ 0)^T, (0\ 1\ 0)^T\}$.Determine geometrically:

(i) L(S), (ii) aff(S), (iii)coni(S), (iv)conv(S)

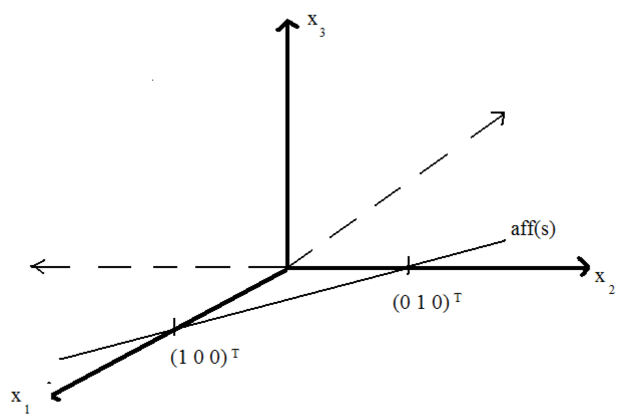
Solution

(i) $L(S) =\{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha,\beta\in R\}=\{(\alpha\ \beta\ 0\ )^T:\alpha,\beta\in R\}$
$=\{(\alpha\ \beta\ 0)^T:\alpha\ \beta\in R\}$
$=x_1x_2$ plane



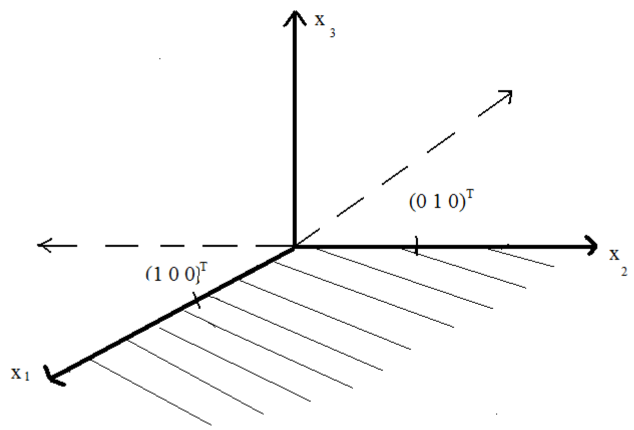
$L(S)=L\{(1\ 0\ 0)^T,(0\ 10)^T\}$ is the x_1x_2 plane

(ii) $Aff(S) = \{\alpha(1\ 0\ 0)^T +\beta(0\ 1\ 0)^T:\alpha,\beta\in R,\alpha+\beta=1\}$
$= \{(\alpha\ \beta\ 0)^T:\alpha,\beta\in R,\beta=1-\alpha\}$
$=\{(\alpha,\ 1-\alpha,\ 0)^T:\alpha\in R\}$

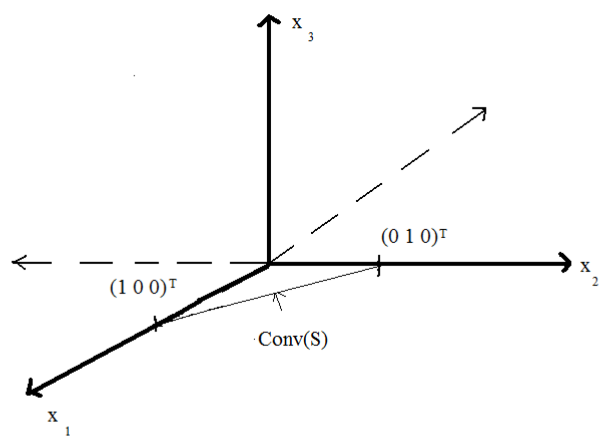


Aff(S) is the line passing through $(1\ 0\ 0)^T$ and $(0\ 1\ 0)^T$.

(iii) $Coni(S) = \{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha\ \beta\in R,\alpha \geq 0,\beta \geq 0\}$
$=\{(\alpha\ \beta\ 0)^T:\alpha \geq 0,\beta \geq 0\}$



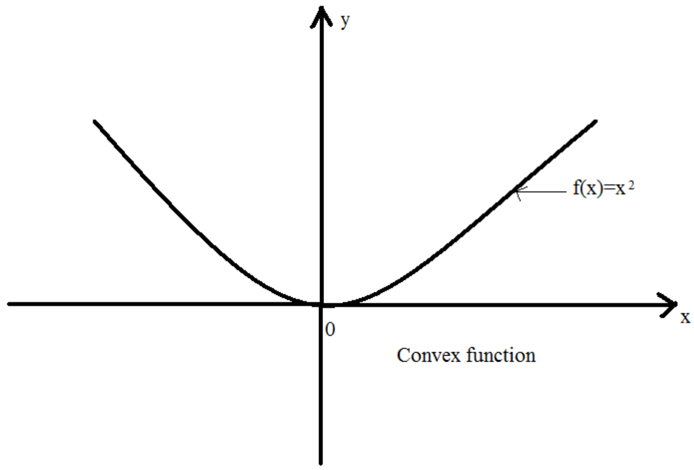
(iv) Conv(S) = $\{\alpha(1\ 0\ 0)^T+\beta(0\ 1\ 0)^T:\alpha,\beta\in R,\alpha \geq 0,\beta \geq 0,\alpha+\beta=1\}$
$=\{(\alpha\ 1\text{-}\alpha\ 0)^T:\alpha\in R,1\geq\alpha\geq 0\}$

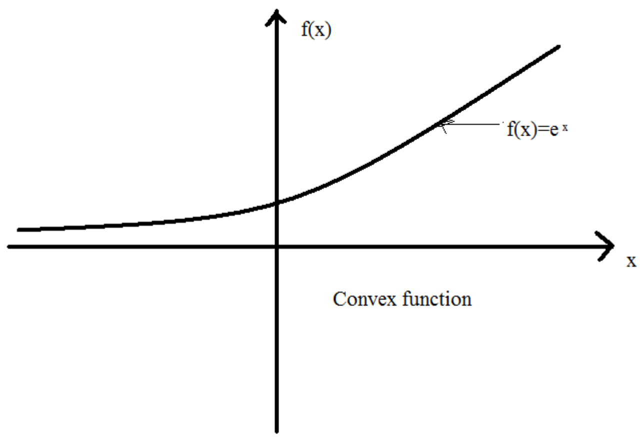


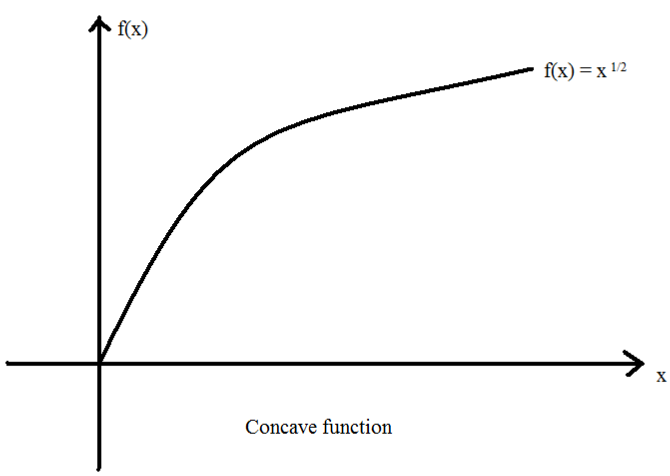
This is a straight line segment that connects the vectors $(1\ 0\ 0)^T$ and $(0\ 1\ 0)^T$

2. For $x\geq 0$, $f(x) = x^2$ and $f(x) = e^x$ are convex functions and $f(x) = x^{1/2}$ is a convex function.

Solution
These facts are evident in the figures below:







3. Show that a linear function of the form $f(x) = ax+b$ is both a convex and a concave function.

Proof

$$f(\alpha x'+(1-\alpha)x'')=a[\alpha x' + (1-\alpha)x'']+b$$
$$=[ax' +b] +(1-\alpha)(ax''+b)$$
$$=\alpha f(x')+(1-\alpha)f(x'')$$

Since both definitions of convex and concave functions are satisfied with equality, $f(x) = ax+b$ is both a convex and a concave function.

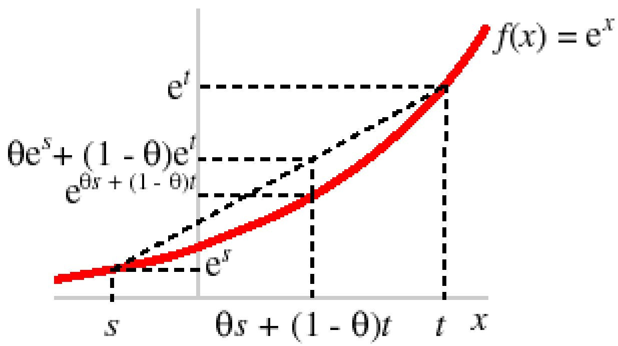
B. Exercises On Convexity And Concavity For Functions Of Several Variable

4.(a) By drawing diagrams, determine which of the following sets is convex.

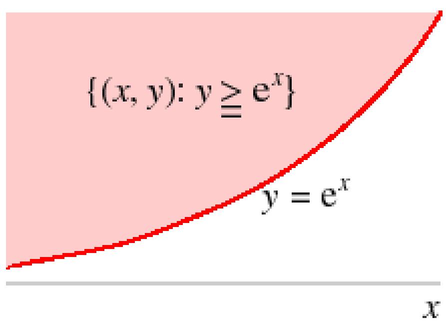
a. $\{(x, y): y = e^x\}$.
b. $\{(x, y): y \geq e^x\}$.
c. $\{(x, y): xy \geq 1, x > 0, y > 0\}$.

Solution

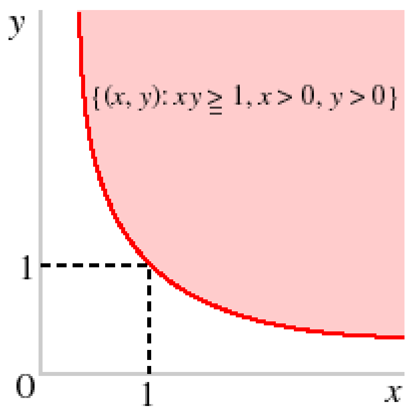
a. Not convex, because $e^{\theta x+(1-\theta)u} \neq \theta e^x + (1-\theta)e^u$, as illustrated in the following figure.



b. Convex, because $e^{\theta x+(1-\theta)u} < \theta e^x + (1-\theta)e^u$ (see the following figure).



c. Convex, because if $xy \geq 1$ and $uv \geq 1$ then $(\theta x + (1-\theta)u)(\theta y + (1-\theta)v) \geq 1$ (see figure).



5. Show that the intersection of two convex sets is convex.

Solution

Let A and B be convex sets. Let $x \in A \cap B$ and $x' \in A \cap B$. We need to show that $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$. Since $x \in A$, $x' \in A$, and A is convex we have $(1 - \lambda)x + \lambda x' \in A$ for all $\lambda \in [0, 1]$. Similarly $(1 - \lambda)x + \lambda x' \in B$ for all $\lambda \in [0, 1]$. Hence $(1 - \lambda)x + \lambda x' \in A \cap B$ for all $\lambda \in [0, 1]$.

1. The function C of many variables and the function D of a single variable are both convex. Define the function f by $f(x,k) = C(x) + D(k)$. Show that f is a convex function (without assuming that C and D are differentiable.

Solution

We have

$$\begin{aligned} f((1-\lambda)(x,k) + \lambda(x',k')) &= C((1-\lambda)x + \lambda x') + D((1-\lambda)k + \lambda k') \\ &\leq (1-\lambda)C(x) + \lambda C(x') + (1-\lambda)D(k) + \lambda D(k') \end{aligned}$$

7. Let $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1 - 2x_2 + 1$. Is f convex, concave, or neither?

Solution

The Hessian matrix of f is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

C. *Exercises On Concave And Convex Functions Of A Single Variable*

8. Show that the function f is convex if and only if the function $-f$ is concave. [Do not assume that the function f is differentiable. The value of the function $-f$ at any point x is $-f(x)$.]

Solution

1. First suppose that the function f is convex. Then for all values of a and b with $a \leq b$ we have

$f((1-\lambda)a + \lambda b) \leq (1-\lambda) f(a) + \lambda f(b)$.

Multiply both sides of this equation by −1 (which changes the inequality):

$-f((1-\lambda)a + \lambda b) \geq -[(1-\lambda) f(a) + \lambda f(b)]$,

or

$-f((1-\lambda)a + \lambda b) \geq (1-\lambda)(-f(a)) + \lambda(-f(b))$.

Thus $-f$ is concave.

Now suppose that the function $-f$ is concave. Then

$$-f((1-\lambda)a + \lambda b) \geq (1-\lambda)(-f(a)) + \lambda(-f(b)).$$

Multiplying both sides of this equation by −1, gives

$f((1-\lambda)a + \lambda b) \leq (1-\lambda) f(a) + \lambda f(b)$,

so that f is convex.

9. The functions f and g are both concave functions of a single variable. Neither function is necessarily differentiable.

a. Is the function h defined by $h(x) = f(x) + g(x)$ necessarily concave, necessarily convex, or not necessarily either?
b. Is the function h defined by $h(x) = -f(x)$ necessarily concave, necessarily convex, or not necessarily either?
c. Is the function $h(x) = f(x)g(x)$ necessarily concave, necessarily convex, or not necessarily either?

Solution

a. We have

$$\begin{aligned} h(\alpha x + (1-\alpha)y) &= f(\alpha x + (1-\alpha)y) + g(\alpha x + (1-a)y) \\ &\geq \alpha f(x) + (1-\alpha) f(y) + \alpha g(x) + (1-\alpha)g(y) \\ &\text{(using the concavity of } f \text{ and of } g) \\ &= \alpha(f(x) + g(x)) + (1-\alpha)(f(y) + g(y)) \\ &= \alpha h(x) + (1-a)h(y). \end{aligned}$$

b. Thus h is necessarily concave.
c. Since f is concave, we have

$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$ for all x, y, and α.

Hence

$-f(\alpha x + (1-\alpha)y) \leq \alpha(-f(x)) + (1-\alpha)(-f(y))$ for all x, y, and α,so that $-f$ is convex.

d. The function h is neither necessarily concave nor necessarily convex. If $f(x) = x$ and $g(x) = x$ then both f and g are concave, but h is convex and not concave. Thus h is not necessarily concave. If $f(x) = x$ and $g(x) = -x$ then both f and g are concave, and h is strictly concave, and hence not convex. Thus h is not necessarily convex.

10. The function $f(x)$ is concave, but not necessarily differentiable. Find the values of the constants a and b for which the function $a f(x) + b$ is concave. (Give a complete argument; no credit for an argument that applies only if f is differentiable.)

Solution

Let $g(x) = a f(x) + b$. Because f is concave we have

$f((1-\alpha)x + \alpha y) \geq (1-\alpha) f(x) + \alpha f(y)$ for all x, y, and $\alpha \in [0,1]$.

Now,

$g((1-\alpha)x + \alpha y) = a f((1-\alpha)x + \alpha y) + b$

and

$(1-\alpha)g(x) + \alpha g(y) = a[(1-\alpha) f(x) + \alpha f(y)] + b$.

Thus

$g((1-\alpha)x + \alpha y) \geq (1-\alpha)g(x) + \alpha g(y)$ for all x, y, and $\alpha \in [0,1]$

if and only if

$a f((1-\alpha)x + \alpha y) \geq a[(1-\alpha) f(x) + \alpha f(y)]$,

or if and only if $a \geq 0$ (using the concavity of f).

11. The function g of a single variable is defined by $g(x) = f(ax + b)$, where f is a concave function of a single variable that is not necessarily differentiable, and a and b are constants with $a \neq 0$. (These constants may be positive or negative.) Either show that the function g is concave, or show that it is not necessarily concave. [Your argument must apply to the case in which f is not necessarily differentiable.]

Solution

We have

$$\begin{aligned} g(\alpha x_1 + (1-\alpha)x_2) &= f(a(\alpha x_1 + (1-\alpha)x_2) + b) \\ &= f(\alpha(ax_1 + b) + (1-\alpha)(ax_2 + b)) \\ &\geq \alpha f(ax_1 + b) + (1-\alpha) f(ax_2 + b) \\ &\quad \text{(by the concavity of } f\text{)} \\ &= \alpha g(x_1) + (1-\alpha)g(x_2). \end{aligned}$$

12. Determine the concavity/convexity of $f(x) = -(1/3)x^2 + 8x - 3$.

Solution

The function is twice-differentiable, because it is a polynomial. We have $f'(x) = -2x/3 + 8$ and $f''(x) = -2/3 < 0$ for all x, so f is strictly concave.

13. Let $f(x) = Ax^{\alpha}$, where $A > 0$ and α are parameters. For what values of α is f (which is twice differentiable) nondecreasing and concave on the interval $[0, \infty)$?

Solution

We have $f'(x) = \alpha A x^{\alpha-1}$ and $f''(x) = \alpha(\alpha - 1)Ax^{\alpha-2}$. For any value of β we have $x^{\beta} \geq 0$ for all $x \geq 0$, so for f to be non decreasing and concave we need $\alpha \geq 0$ and $\alpha(\alpha - 1) \leq 0$, or equivalently $0 \leq \alpha \leq 1$.

14. Find numbers a and b such that the graph of the function $f(x) = ax^3 + bx^2$ passes through $(-1, 1)$ and has an inflection point at $x = 1/2$.

Solution

For the graph of the function to pass through $(-1,1)$ we need $f(-1) = 1$, which implies that $-a + b = 1$. Now, we have $f'(x) = 3ax^2 + 2bx$ and $f''(x) = 6ax + 2b$, so for f to have an inflection point at 1/2 we need

$f''(1/2) = 0$, which yields $3a + 2b = 0$. Solving these two equations in a and b yields $a = -2/5$, $b = 3/5$.

IV. CONCLUSIONS

Convex and concave functions can be applicable in many sectors of life. From our work so far, we can see that convex and concave functions can be applied in solving problems from management, economics and in fact our everyday life. In analyzing graphs, the idea of convexity is used. Calculus also makes use of convex functions. Some of the most important applications of calculus require the use of the derivative of finding the maxima and minima. If we have functions that model cost, revenue, or population growth, for example, we can apply the methods of calculus to find the minima and maxima of the function. We have realized that a function when differentiated twice will give you a minima or a maxima.

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